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Common Fixed Point Theorems for Six Self Maps in FM-Spaces Using Common Limit in Range Concerning Two Pairs of Products of Two Different Self-maps

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Abstract
In this article, we do a study of common fixed point theorems for six self-maps in FM-Spaces using common limit in range property concerning two pairs of products of two different self-maps. We use the properties (CLRTH) and (CLRSR) along with contractive type implicit relations to prove our results. In support of our result, an example has been provided. Our findings are like those of Kumar and Chouhan [12]. Kumar and Chauhan demonstrated their primary result in [12] by improving and generalizing Aalam, Kumar, and Pants' [1] results. In past, many authors have done study of common fixed point using (E-A) property (like Aalam et. al. [1] proved results using this property), and then these results were improved and generalized by using common (E-A) property as this property is superior to (E-A) property, as the closeness of subspace is required to prove a required result on common fixed point by using these properties, which is a drawback. We improve and generalize all results on these properties using common limit in range property. The goal of this note is to refine and generalize Kumar and Chauhan's [12] results on a common fixed point, as well as some earlier comparable results.

Key-words: Fuzzy Metric Spaces (FM-spaces), Common Fixed Point (CFP), Weak Compatible Maps (WCM), Implicit Relations, CLR Property.
MSC: 54H25, 47H10.

1. Introduction

In 1965, Zadeh [23] proposed fuzzy sets, and Kramosil and Michalek [10] proposed FM-spaces in 1975. The contraction principle was then proven in the context of FM-spaces by Grabiec [6]. As a result, George and Veeramani [5] used continuous t-norm to revise the design of FM-spaces.

After Jungck [8] pioneered the idea of compatible maps in metric space, Mishra et al. [15] developed it further in 1994 as asymptotically commuting maps. After Jungck [9] pioneered the idea of WCM in the context of metric space, Singh et al. [20] investigated it in the aspect of FM-spaces.

Pant et al. [17] prolonged the study of the CFP of a pair of non-compatible maps (which had previously been studied in metric space by Pant [16]) and the (E-A) property to FM-spaces in 2007. In 2002, A.Aamri and E. Moutawakil [2] investigated a new property (E-A) for a pair of self maps that broadens the notion of non-compatible maps in metric spaces. Many results in FM-spaces have been obtained by employing property (E-A) (refer [1],[3],[11], and [14]).

Imdad et al. [7] in 2009, suggested the concept of pairwise commuting mappings (PCM). Implicit relation has been studied as a potential new tool for locating the CFP of contraction maps. Aalam et al. [1] exhibited a famous fixed point theorem in FM-spaces, which is a broad statement of Singh's et al. [20] result, without accounting for space completeness or continuity of involved mappings. Following that, Kumar and Chauhan [12] used contractive type implicit relations to extend the research on a CFP theorem in FM–spaces to six self-maps (as studied by Aalam et al. [1] in FM-Spaces), as well as four families of mappings in FM-Spaces.

Sintunavarat, W and Kumam, P [21] recently introduced a new notion in FM-spaces called a common limit range property, or property CLR. It should be observed that property CLR does not require the subspace's closeness condition, whereas property (E-A) does for the emergence of the fixed point in FM-spaces. As a outcome, researchers are now focusing on this property CLR to generalize or improve previous findings.

The primary aim of this study is to use the properties (CLRTH) and (CLRSR) as well as implicit relation to establishing a CFP theorem for six self maps in FM-Spaces.

2. Prelims

(Following definitions are required to derive our results.)

Def. 2.1. Fuzzy Set **[23], Def. 2.2.** Continuous t-norm **[19]**

As above definitions 2.1 and 2.2 are basics and already defined in the literature so we give only the references of them.

Def. 2.3. [10] A set $(X, M, *)$ is termed a FM-Space if $*$ is a continuous t-norm, X is an arbitrary nonempty set, and M is a fuzzy set on $X^2 \times [0, \infty)$ that satisfy a set of circumstances $\forall\ \beta, \gamma, \delta\ \in\ X$ and,$\tau, \omega > 0$:

(FM-1) $M(\beta,\gamma,0) = 0$;

(FM-2) $M(\beta,\gamma,\tau) = 1 \ \forall \ \tau > 0 \ \Leftrightarrow \ \beta \ = \ \gamma$;

(FM-3) $M(\beta,\gamma,\tau) = M(\gamma,\beta,\tau)$;

(FM-4) $M(\beta,\gamma,\tau) \ * M(\gamma,\delta,s) \ \leq \ M(\beta,\gamma,\tau \ + \ \omega)$;

(FM-5) $M(\beta,\gamma,.)$ is left continuous from $[0,\infty)$to $[0,1]$

(FM-6) $\lim_{n\to\infty} M(\beta,\gamma,\tau) = \ 1$

Example 2.4. [5] Consider (X,d) be a metric space. Let $\rho * \sigma = \min\{\rho,\sigma\} \ \forall \ \rho,\sigma \in X$ and $\tau > 0$, $M(\beta,\gamma,\tau) = \frac{\tau}{\tau + d(\beta,\gamma)}$. Then $(X,M,*)$ is a FM-Space.

Above example shows that every metric prompts a fuzzy metric.

Def. 2.5. [6] Let (X, M, *) be a FM-Space.

(1) if for a sequence $\{x_n\}$ in FM-Space, $lim_{n\to\infty} M(x_n,x,t) = 1 \ \forall$ t > 0, then it is said to be convergent.

(2) if $lim_{n\to\infty}M(x_{n+r},x_n,t) = 1 \ \forall \ t > 0$ and $r > 0$, then it's referred to as Cauchy.

(3) if every Cauchy sequence in X converges in X, then X is regarded to be a complete FM-Space.

Lemma 2.6. [6] $\forall$, $x,y \ \in X, M(x,y,.)$ is non-decreasing.

Lemma 2.7. [13] Let $M(x,y,*)$ be a FM- Space. Then M is a continuous on $X^2 \times (0, \infty)$.

Def. 2.8. [15] Assume A, S are mappings from FM-space (X, M, *) to itself. If $\forall t$, $lim_{n\to\infty}M(ASx_n,SAx_n,t) = 1$ and the sequence $\{x_n\}$ in X is such that $lim_{n\to\infty}Ax_n = z = lim_{n\to\infty}Sx_n$ for some $z \in X$, then A and S are known as compatible maps.

Def. 2.9. [21] Assume A and S are two self-maps.

If $Az = Sz$ implies that $ASz = SAz$, then A and S are known as weakly compatible maps.

Def. 2.10. [17] Assume that A and S are two FM self-maps. If X contains at least one sequence $\{x_n\}$ such that $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = \ z$ for some $z \in X$, the maps A and S, then, entertain the property (E-A).

It's worth noting that, weak compatibility of the maps and property (E-A) are unrelated (refer [18], Example 2.2).

According to Def. 2.10, self-maps A,S in a FM-Space are incompatible iff X contains at least one sequence$\{x_n\}$ such that $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = z$ for some z in X, but for some $t > 0$, either $lim_{n\to\infty} \ M(ASx_n,SAx_n,t) \neq 1$ or the limit doesn't exist. As a result, any two non-compatible self-maps share the property (E-A) defined in Def. 2.10. Maps which satisfy property (E-A), on the other hand, do not have to be incompatible (refer [4], Example 1).

Def. 2.11. [21, 22] In a FM-Spaces (X,M,*), a pair of self-maps (S, T) is said to entertain the (CLR_T) property concerning mapping T if occurs $\{x_n\}$ in X such that $lim_{n\to\infty}\, S\, x_n = lim_{n\to\infty}\, Tx_n = Tu$ for some u in X.

Def. 2.12. [12**]** Consider Φ is the collection of all functions which are real and continuous. And is defined as $\phi : (R^+)^4 \to R$, which is non-decreasing with regard to first argument and satisfies the set of circumstances:

(A1) For $\zeta,\eta \geq 0, \phi(\zeta,\eta,\zeta,\eta) \geq 0$ or $\phi\,(\zeta,\eta,\eta,\zeta) \geq 0$ implies that $\zeta \geq \eta$.

(A2) $\phi(\zeta,\zeta,1,1) \geq 0$ implies that $\zeta \geq 1$.

Ex. 2.13. [12] Consider $\phi(t_1, t_2, t_3, t_4) = a\, t_1 + bt_2 + c\, t_3 + dt_4$, where a, b, c and d are real numbers. If $a > max\,\{b,d\}$, $a + c = b + d > 0$, then $\phi \in \Phi$.

3. Principal Results

Kumar and Chauhan [12] established the following result:

Theorem 3.1. Let F, G, R, S, H, T be self-maps of a FM-space $(X, M,*)$ satisfying

(3.1) (F, SR) or (G, TH) satisfies the property (E. A);

(3.2)$\phi(M\,(Fx, Gy, r), M(SRx, THy, r), M(Fx, SRx, r), M(Gy, THy, r)) \geq 0, \forall r > 0,\ x, y \in X$ and for some $\phi \in \Phi$;

(3.3) $F(X) \subseteq TH(X), G(X) \subseteq SR(X)$;

(3.4) One of $F(X), G(X), SR(X), TH(X)$ is a complete subspace of X;

Then (F, SR), (G, TH) have a coincidence point.

Furthermore, F, G, R, S, H , T have a unique CFP, when (F, SR) and (G, TH) commute pairwise.

Now we generalize and improve Theorem-3.1 as follows:

Theorem 3.2.

Let $(X, M,*)$ be a FM-Space with $\rho * \sigma = \min\{\rho,\sigma\}$.Let A, B, R, S, H, T be self-maps of a FM-Space X that satisfy (3.2) with the set of circumstances:

(3.5) $B(X) \subseteq SR(X)$ and (B, TH) satisfies the property (CLR_{TH})

or

$A(X) \subseteq TH(X)$ and (A, SR) satisfies the property (CLR_{SR})

(3.6) (A, SR), (B, TH) are weakly compatible.

Then (A, SR), (B, TH) have a coincidence point. Furthermore, A, B, R, S, H, T have a unique CFP in X.

Proof. Suppose $B(X) \subseteq SR(X)$, and (B, TH) meets (CLR_{TH}) property so exists $\{x_n\}$ in X s.t. $Bx_n \to THx, THx_n \to THx$ for some $x \in X$ as $n \to \infty$.

As, $B(X) \subseteq SR(X)$ so occurs $\{y_n\}$ in X such that $Bx_n = SRy_n$. Hence, $SRy_n \to THx$ as $n \to \infty$.

Now we show that $Ay_n \to THx$ as $n \to \infty$. By putting $x = y_n$ and $y = x_n$ in (3.2), We have

$\phi(M(Ay_n, Bx_n, t), M(SRy_n, THx_n, t), M(Ay_n, SRy_n, t), M(Bx_n, TH, t)) \geq 0$, $\forall\, t > 0, x, y \in X$ and for some ϕ∈Φ.

$\phi(M(Ay_n, THx, t), M(THx, THx, t), M(Ay_n, THx, t), M(THx, THx, t)) \geq 0$

$\phi(M(Ay_n, THx, t), 1, M(Ay_n, THx, t), 1) \geq 0$

By using A_2 we get $M(Ay_n, THx, t) \geq 1$

Hence, $M(Ay_n, THx, t) = 1$

i.e $lim_{n\to\infty} Ay_n = THx$ or $z = THx$

Subsequently we have Bx_n, THx_n, SRy_n, and Ay_n converges to z.

Now we shall show that $Bx = z = lim_{n\to\infty} Ay_n$

Taking $x = y_n$ and $y = x$ in (3.2)

$\phi(M(Ay_n, Bx, t), M(SRy_n, THx, t), M(Ay_n, SRy_n, t), M(Bx, THx, t)) \geq 0$

$\phi(M(Ay_n, Bx, t), 1, 1, M(Bx, Ay_n, t)) \geq 0$

By using A_2 we get $M(Ay_n, Bx, t) \geq 1$

Hence $Bx = lim_{n\to\infty} Ay_n$ or $z = Bx$

Finally, $z = THx = Bx$ or $z = Bx = THx$

Since (B, TH) is weakly compatible so $Bz = THz$ ... (i)

This demonstrates that (B, TH) has a coincidence point.

As, $B(X) \subseteq SR(X)$, so exists some y in X such that $Bx = SRy = z$

Next, we demonstrate that $SRy = Ay = z$

Taking $y = x_n$ and $x = y$ in (3.2), we have

$\phi(M(Ay, Bx_n, t), M(SRy, THx_n, t), M(Ay, SRy, t), M(Bx_n, THx_n, t)) \geq 0$

$\phi(M(Ay, z, t), 1, M(Ay, z, t), 1) \geq 0$

By using A_2 we get $Ay = z = SRy$

But the pair (A, SR) is weakly compatible so $Az = SRz$ ... (ii)

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This indicates that (A, SR) has a coincidence point.

Next, we claim that $Az = Bz$

Taking $x = z$ and $y = z$ in (3.2), we have

$\phi(M(Az, Bz, t), M(SRz, THz, t), M(Az, SRz, t), M(Bz, THz, t)) \geq 0$

$\phi(M(Az, Bz, t), M(Az, Bz, t), 1, 1) \geq 0$

By using A_2 we get $Az = Bz$ ... (iii)

Therefore, by (i), (ii) and (iii) we get

$Az = Bz = THz = SRz = z$ ... (iv)

Next, we prove that z is a fixed point of A, S, and R.

On setting $x = Rz$, $y = z$ in (3.2) we get

$\phi(M(A(Rz), Bz, t), M(SR(Rz), THz, t), M(A(Rz), SR(Rz), t), M(Bz, THz, t)) \geq 0$

$\phi(M(Rz, z, t), M(Rz, z, t), 1, 1) \geq 0$

Again, by using A_2 we get, $Rz = z$

Since $Sz = S(Rz) = SRz = z$

This shows that $z = Sz = Az = Rz$ ... (v)

Again, on taking, $x = z, y = Hz$ in (3.2) and then solving, we have $Hz = z$

Since $T(z) = T(Hz) = z$

Implies that $z = Bz = Tz = Hz$ ... (vi)

Thus, by $(iv), (v)$, and (vi) z is a common fixed point of all the six mappings.

For uniqueness: let w and z are two distinct common fixed point. By taking $x = z$, $y = w$ in (3.2) we get $z = w$. This shows z is unique CFP.

On setting $R = H = IX$ in Th.-3.2, we get the below result:

Corollary 3.2. Suppose that A, B, S, and T are self-maps in FM-space that satisfy

(3.21)$B(X) \subseteq S(X)$, and (B, T) satisfies the property (CLR_T) or

$A(X) \subseteq T(X)$, and (A, S) satisfies the property (CLR_S)

(3.22) (A, S), (B, T) are weakly compatible

(3.23) $\phi(M(Ax, By, t), M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t)) \geq 0, for\ all\ t > 0, x, y \in X$ and for some $\phi \in \Phi$.

Then (A, S), (B, T) have a coincidence point. Furthermore, A, B, S, T have a unique CFP in X.

Proof. This corollary's proof is analogous to that of Theorem 3.2, so it is omitted.

On setting $A = B$ and $S = T$ in above Coro.-3.2, the below result is obtained:

Corollary 3.3. Suppose that A, S are self-maps of a FM-space that satisfy.

(3.31) $A(X) \subseteq T(X)$, and (A, S) satisfies the property (CLR_S).

(3.32) (A, S) is weakly compatible +

(3.33) $\phi(M(Ax, Ay, t), M(Sx, Sy, t), M(Ax, Sx, t), M(Ay, Sy, t)) \geq 0$, for all $t > 0, x, y \in X$ and for some $\phi \in \Phi$.

Then (A, S) has a coincidence point. In addition, A, S have a unique CFP.

Example 3.4. Let $(X, M,*)$ be a FM-Space with $X = [0,1]$, a t-norm $*$ be described as $\rho * \sigma = \min\{\rho, \sigma\}$ for all ρ, σ in $[0,1]$ and M be a fuzzy set on $X^2 \times (0, \infty)$ which is described as $M(x, y, t) = [\exp(\frac{|x-y|}{t})]^{-1}$ for all x, y in X and $t > 0$.

Let $\emptyset: (R^+)^4 \to R$ be considered as in example 2.13.

Consider $A, B, R, S, H, T: X \to X$ by $Ax = \frac{x}{27}, Bx = \frac{x}{9}, Sx = \frac{x}{3}, Rx = x, Hx = \frac{2x}{3}, Tx = \frac{3x}{2}$ respectively.

Then for all x, y in X and > 0, we have

$$M(Ax, By, t) = [\exp(\frac{\left|\frac{x}{27} - \frac{y}{9}\right|}{t})]^{-1}$$

$$\geq [\exp(\frac{\left|\frac{x}{3} - y\right|}{t})]^{-1} = M(SRx, THy, t)$$

Which shows that, $\phi(M\ (Ax, By, t), M(SRx, THy, t), M(Ax, SRx, t), M(By, THy, t)) \geq 0$ for some $\phi \in \Phi$.

As a result, Theorem-3.2's condition 3.2 has been met.

Further, $B(X) = \left[0, \frac{1}{9}\right] \subseteq \left[0, \frac{1}{3}\right] = SR(X)$ and $A(X) = \left[0, \frac{1}{27}\right] \subseteq [0,1] = TH(X)$. Considering the sequence $\{x_n\} = \{\frac{1}{n}\}$ so that $lim_{n\to\infty} Bx_n = lim_{n\to\infty} THx_n = 0$, hence (B, TH) fulfills the property CLR_{TH}. Similarly, (A, SR) fulfills the property CLR_{SR}. Also, (A, S), (B, T) are weakly compatible.

Hence, all of Theorem-3.2's criteria are met, and 0 is unique CFP of all maps.

4. Conclusions

The property (E-A) is discovered to buy range containment without requiring continuity, as well as reducing the maps' commutativity conditions to commutativity at their points of coincidence. In addition, the property (E-A) enables the whole-space completeness requirement to be replaced with a more natural range-space completeness condition. As a result of integrating the concept of property (E-A), Kumar and Chauhan improved many of the results in [12]. As an improvement/generalization of a result, we proved our main result (Th-3.2) by using the notion of property CLR (concerning mappings TH and SR), in this property the condition of closeness of range subspaces is not required. As a result, in this paper, we proved Theorem-3.2 and obtained the same result without assuming the completeness of any X subspaces and by relaxing many of the conditions in Theorem-3.1. In [12], Theorem-3.1 was proven as a fixed point theorem under stronger contractive conditions, whereas we proved the same result under weaker contractive conditions. Exercising Theorem-3.2's validity is proven in Example-3.4.

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