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Convergence and Analysis of K-Iteration Scheme in Complex Valued Banach Space

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Abstract
In this paper, the convergence of a three-step iterative process i.e., K-iteration scheme is established in complex valued Banach spaces under the setting of generalized F- contractions. Also, the convergence of Picard Ishikawa hybrid process is established with same conditions. This work hence extends the convergence of three step iterative schemes in complex valued Banach spaces and further enhances the scope of research and analysis of the work in this area.

Key-words: Three-Step Iteration, Banach Space, Complex Valued Banach Space, Generalized F-contractions.

1. Introduction and Preliminaries

The fixed point theory has been one of the most interesting and critically significant concept in various applied areas of science apart from some other areas like economics, optimization problems, control theory and even medical field. The direct application of fixed point theory involves finding the solution of differential equations and integral equations. These topics are directly linked to real life situations and hence fixed point theory holds an important role in their solutions. The major revolution in this regard came into existence in 1922 with Banach contraction principle [2]. The Banach contraction principle used a general iteration scheme in complete metric spaces to prove a fixed point theorem and its uniqueness using a linear contraction. The results of Banach have been generalized, extended and improved over the years by many researchers thereby enhancing the scope of study in fixed point theory. These generalizations have been made by modifying the conditions involved in obtaining the results. This includes the framing and use of various iterative schemes, changing the contractive conditions or utilizing different approaches for solution. Some of the well-known extensions of metric spaces include pseudo metric spaces, partial metric spaces, G-metric spaces and cone metric spaces etc.

Another important aspect of establishing a fixed point result is an iterative scheme. The iterative schemes play a crucial role in establishing the fixed point theory and over the years, a number of iterative schemes have been defined. These have been extending and improving the structure of fixed point theory (See [1], [3], [4], [5], [9]). This paper mainly deals with the convergence of fixed point iteration methods. The literature gives an insight about different iteration schemes which are convergent under some set of contraction mappings but not for others. Hence, it is necessary to determine whether the iteration process chosen for a particular setting of contraction mappings is convergent to its fixed point or not. Some of the iterative schemes are defined as follows:

Introduced in 1953, Mann iteration scheme [3] was developed to prove the convergence of a sequence of real numbers in Banach principle.

Definition 1.1 [3] If E is a normed linear space and $\mathrm{T}: \mathrm{E} \rightarrow \mathrm{E}$ be a self-mapping. Define an arbitrary point $x_0 \in$ E and a real number sequence by $\{\alpha_n\} \in [0, 1]$. Then the scheme defined by,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \text{ where } n \in \mathbb{N} \cup \{0\}$$

is referred to as Mann Iterative scheme or Mann Iteration.

Mann iteration scheme failed to converge for Lipschitzian pseudo contractive maps and hence, this motivated the need of modification in Mann Iteration and gave rise to Ishikawa iteration [5] scheme that came into existence in 1974.

Definition 1.2 [5] If E is a normed linear space and $\mathrm{T}: \mathrm{E} \rightarrow \mathrm{E}$ is a self-mapping. Define the sequences of positive real numbers represented by $\{\alpha_n\}$ and $\{\beta_n\} \in [0, 1]$. Then the scheme defined by,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n,$$

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n, \text{ where } n \in \mathbb{N} \cup \{0\}$$

is referred to as Ishikawa Iterative scheme or Ishikawa Iteration.

A modification in the Mann iteration scheme was developed by replacing α_n with a constant λ and the new scheme so created is known as Krasnoselski Iteration [4] given by:

$$x_{n+1} = (1 - \lambda)x_n + \lambda \mathrm{T} x_n, \text{ where } n \in \mathbb{N} \cup \{0\}$$

When $\alpha_n = 1$ in Mann Iteration, the scheme so developed is referred to as Picard Iteration scheme [1] given by,

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$$x_{n+1} = \mathrm{T}x_n, \text{ where } n \in \mathbb{N} \cup \{0\}$$

The Picard Iterative scheme is also referred to as the method of successive approximations.

Since Picard iterative scheme failed to converge for some operators so, to determine the fixed point for those operators, the Krasnoselski, Mann and Ishikawa iteration procedures were developed.

Over the time, with some other modifications in the already existing iterative processes, have also been defined.

Further, the single-step iteration (Picard), two-step iterations (Mann and Ishikawa) were extended to three-step iteration defined by Noor et al [9] in 2000 and the new scheme was called as Noor iteration scheme.

Recently, in 2018, a three-step iteration process was defined by N. Hussain et al [29] which is claimed to be the fastest converging iterative scheme by the authors as compared with some of the existing iterations. Named as K-iterative process it is defined as follows:

Definition 1.3 [29] For $x_0 \in \mathsf{C}$, the K-iteration process is given as:

$$x_{n+1} = \mathcal{T}y_n$$
$$y_n = \mathcal{T}\{(1-\alpha_n)\mathcal{T}x_n + \alpha_n\mathcal{T}z_n\}$$
$$z_n = (1-\beta_n)x_n + \beta_n\mathcal{T}x_n,$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ are real sequences in [0, 1].

Also, in 2019, G. Okeke[31] defined Picard Ishikawa hybrid process and analyzed the convergence of this scheme with some standard iterative processes to prove that it is converging faster as compared to them in the setting of complex valued Banach space. Also, the author utilized this new iterative process to find the solution of some delay differential equations.

Definition 1.4 [31] For $n \in \mathbb{N}$ *and* $u_1 \in \mathsf{C}$, the Picard Ishikawa Hybrid process is given as:

$$u_{n+1} = \mathcal{T}v_n$$
$$v_n = (1-\alpha_n)u_n + \alpha_n\mathcal{T}w_n$$
$$w_n = (1-\beta_n)u_n + \beta_n\mathcal{T}u_n,$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ are real sequences in (0, 1).

Our paper deals mainly with these two iterative schemes, their convergence and analysis.

Another fundamental criterion for fixed point results is the contractive condition. The concept of contraction was introduced by Wardowski[15] in 2012. This result was a generalization of Banach contraction principle and the author also proved fixed point theorem for the same. The concept was firther extended to metric spaces for establishing some fixed point results in them by Abbas et al [16]. Later, the idea of F-weak contraction was given by Wardowski and Van Dung [23]. Further Husain et al [21] proved fixed point results with the help of α-GF contraction in ordered metric space. The idea of F-contraction was extended to graphs by Batra et al [24]. In the year 2003, a new type of contractive like mappings were defined by Imoru and Olatinwo [10] where the authors used it for establishing some stability results which is defined as follows:

Definition 1.5 [10]: Let E represents a normed space, C be a non-empty convex subset of a normed space E. Let $\mathcal{T}$ be a self-map of C, then the class of operators called contractive like mappings are given as:

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \varphi(\|x - \mathcal{T}x\|) + \delta\|x - y\|$$

for all $x, y \in E$ where $\delta \in [0,1)$ and $\varphi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ denotes a monotonically increasing function such that $\varphi(0) = 0$

Lemma 1.6 [11]: If ζ be a real number such that $0 \leq \zeta < 1$ and a sequence of positive numbers be given as $\{\psi_n\}$ where $n \in \mathbb{N}$. Also, $\lim_{n\longrightarrow\infty} \psi_n = 0$, then for any sequence of positive numbers, $\{\mathfrak{y}_n\}$ which satisfies the condition $\mathfrak{y}_n \leq \zeta\mathfrak{y}_n + \psi_n$ for $n \geq 1$ we have,

$$\lim_{n\longrightarrow\infty} \mathfrak{y}_n = 0$$

It is also interesting to note that the although a number of problems can be solved with metric spaces but generalized Banach spaces analyze them more accurately. A series of results in this regard has been proved by Argyros et al. [27], Traub [6] and Meyer [7].

In 2011, complex valued metric space came into existence when it was defined by Azam et al. [13] who established few results in fixed point utilizing a pair of mappings which satisfied some rational inequalities. These rational expressions were meaningless in cone metric spaces and hence the theorems suggested by Azam et al [13] proved to be very significant. Complex valued metric spaces found extensive significance in pure as well as applied mathematics. The results of complex valued metric spaces have been improved and extended a lot by many researchers so far.

In 2019, the notion of complex valued metric spaces was generalized to complex valued Banach spaces by Okeke [30] who gave approximation of iterative schemes in them.

Definition 1.7 [13] Consider that $\mathbb{C}$ represents a complex number set and let ω_1, $\omega_2 \in \mathbb{C}$. A partial order represented by $' \leq '$ defined on $\mathbb{C}$ is given as follows:

$$\omega_1 \leq \omega_2 \text{ if and only if } Re(\omega_1) \leq Re(\omega_2) \text{ and } Im(\omega_1) \leq Im(\omega_2)$$

Consequently,

i. $Re(\omega_1) = Re(\omega_2), Im(\omega_1) < Im(\omega_2)$
ii. $Re(\omega_1) < Re(\omega_2), Im(\omega_1) = Im(\omega_2)$
iii. $Re(\omega_1) < Re(\omega_2), Im(\omega_1) < Im(\omega_2)$

iv. $Re(\omega_1) = Re(\omega_2), Im(\omega_1) = Im(\omega_2)$

Definition 1.8 [13] Let X denotes a non-empty subset of $\mathbb{C}$. Define a mapping $d: X \to X$ such that:

i. $0 \leq d(\alpha, \beta)\ \forall\ \alpha, \beta \in X$ *and* $d(\alpha, \beta) = 0$ *if and if only if* $\alpha = \beta$.

ii. $d(\alpha, \beta) = d(\beta, \alpha)\ \forall\ \alpha, \beta \in X$

iii. $d(\alpha, \beta) \leq d(\alpha, \gamma) + d(\gamma, \beta)\ \forall\ \alpha, \beta, \gamma \in X$

Then d is referred to as a complex valued metric on X and (X, d) is referred to as a complex valued metric space.

Definition 1.9 [30] Assume that E is a linear space defined on a field represented by $\mathbb{K}$ where $\mathbb{K}$ can be a real number set or a complex number set. Then, a complex valued operator, $\|.\|$ such that $\|.\|: \mathrm{E} \to \mathbb{C}$ is said to be complex valued norm on E if it satisfies the properties as follows:

i) $\|\omega\| = 0$ *if and only if* $\omega = 0, \omega \in \mathrm{E}$

ii) $\|k\omega\| = |k|.\|\omega\|\ \forall\ k \in \mathbb{K},\ \omega \in \mathrm{E}$

iii) $\|\omega_1 + \omega_2\| \leq \|\omega_1\| + \|\omega_2\|\ \forall \omega_1, \omega_2 \in \mathrm{E}$

Further, a linear space along with a complex valued norm that has been defined over the same space is known as a complex valued normed linear space and it is represented by $(\mathrm{E}, \|.\|)$

Now, suppose a sequence $\{x_n\}$ is defined over E and some $x \in \mathrm{E}$. If, for some $c \in \mathbb{C}$ such that $c > 0$, $\exists$ some $n_0 \in \mathbb{N}$ such that $\forall\ n > n_0$, if $\|x_n - x_{n+m}\| < c$, then the sequence $\{x_n\}$ is referred to as a Cauchy sequence in $(\mathrm{E}, \|.\|)$

Also, when each Cauchy sequence defined over $(\mathrm{E}, \|.\|)$ also converges in $(\mathrm{E}, \|.\|)$, then the space $(\mathrm{E}, \|.\|)$ is referred to as Complex valued Banach space [11].

2. Convergence Result for Three Step Iterative Schemes for Complex Valued Banach Space

In this section we prove the convergence of the K-Iteration process in a complex valued Banach space for generalized F-contractions.

Theorem 2.1: Let $\mathcal{D}$ represents a convex subset of a complex valued Banach space $(\mathrm{E}, \|.\|)$. Let $\mathcal{T}: \mathcal{D} \to \mathcal{D}$ be a linear mapping which satisfies the condition (1.5). Let the sequence $\{x_n\}$ represents the K-iterative process in $\mathcal{D}$ and defined as follows:

$$x_{n+1} = \mathcal{T} y_n$$

$$y_n = \mathcal{T}\{(1 - \alpha_n)\mathcal{T} x_n + \alpha_n \mathcal{T} z_n\}$$

$$z_n = (1 - \beta_n) x_n + \beta_n \mathcal{T} x_n$$

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Where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0, 1]. Then, the sequence $\{x_n\}$ is convergent to the fixed point of $\mathcal{T}$.

Proof: From the contractive condition it can be easily obtained that $\mathcal{T}$ has a fixed point and let that point be given by x^*. To prove that the sequence $\{x_n\}$ converges to the fixed point x^* consider,

$$\|x_{n+1} - x^*\| = \|\mathcal{T}y_n - x^*\|$$
$$\leq \varphi(\|y_n - \mathcal{T}y_n\|) + \delta\|y_n - x^*\|$$
$$\leq \varphi(\|x^* - \mathcal{T}x^*\|) + \delta\|y_n - x^*\| \qquad (2.1)$$

Now consider,

$$\|y_n - x^*\| = \|\mathcal{T}(1-\alpha_n)\mathcal{T}x_n + \alpha_n\mathcal{T}z_n - x^*\|$$
$$\leq \varphi(\|(1-\alpha_n)\mathcal{T}x_n + \alpha_n\mathcal{T}z_n - \mathcal{T}((1-\alpha_n)\mathcal{T}x_n + \alpha_n\mathcal{T}z_n)\|) + \delta\|(1-\alpha_n)\mathcal{T}x_n + \alpha_n\mathcal{T}z_n - x^*\|$$

Further,

$$\|y_n - x^*\| \leq \varphi(\|(1-\alpha_n)\mathcal{T}x_n + \alpha_n\mathcal{T}z_n - (1-\alpha_n)\mathcal{T}^2x_n + \alpha_n\mathcal{T}^2z_n\|)$$
$$+ \delta\|(1-\alpha_n)(\mathcal{T}x_n - x^*) + \alpha_n(\mathcal{T}z_n - x^*)\|$$
$$\leq \varphi(\|(1-\alpha_n)\mathcal{T}(x_n - \mathcal{T}x_n) + \alpha_n\mathcal{T}(z_n - \mathcal{T}z_n)\|) + \delta\|(1-\alpha_n)(\mathcal{T}x_n - x^*) + \alpha_n(\mathcal{T}z_n - x^*)\|$$
$$\leq \varphi(\|(1-\alpha_n)\mathcal{T}(x^* - \mathcal{T}x^*) + \alpha_n\mathcal{T}(x^* - \mathcal{T}x^*)\|) + \delta\|(1-\alpha_n)(\mathcal{T}x_n - x^*)||+\delta\alpha_n||(\mathcal{T}z_n - x^*)\|$$
$$(2.2)$$

Now consider,

$$\|\mathcal{T}z_n - x^*\| = \left\|\mathcal{T}\left((1-\beta_n)x_n + \beta_n\mathcal{T}x_n\right) - x^*\right\|$$
$$\leq \varphi\left\|(1-\beta_n)x_n + \beta_n\mathcal{T}x_n - \mathcal{T}\left((1-\beta_n)x_n + \beta_n\mathcal{T}x_n\right)\right\| + \delta\|(1-\beta_n)x_n + \beta_n\mathcal{T}x_n - x^*\|$$
$$\leq \varphi\|(1-\beta_n)(x_n - \mathcal{T}x_n) + \beta_n\mathcal{T}(x_n - \mathcal{T}x_n)\| + \delta\|(1-\beta_n)(x_n - x^*) + \beta_n(\mathcal{T}x_n - x^*)\|$$
$$\leq \varphi(\|(1-\beta_n)(x^* - \mathcal{T}x^*) + \beta_n\mathcal{T}(x^* - \mathcal{T}x^*)\|) + \delta\|(1-\beta_n)(x_n - x^*)||+\delta\beta_n||(\mathcal{T}x_n - x^*)\|$$

Hence eqn. (2) becomes,

$$\|y_n - x^*\| \ \leq \varphi(\|(1-\alpha_n)\mathcal{T}(x^* - \mathcal{T}x^*) + \alpha_n\mathcal{T}(x^* - \mathcal{T}x^*)\|)$$
$$+ \delta\alpha_n(\varphi(\|(1-\beta_n)(x^* - \mathcal{T}x^*) + \beta_n\mathcal{T}(x^* - \mathcal{T}x^*)\|) + \delta(1-\alpha_n)\|\mathcal{T}x_n - x^*\|$$
$$+ \delta\beta_n\|\mathcal{T}x_n - x^*\| + \delta(1-\beta_n)||x_n - x^*||$$
$$\leq (1-\alpha_n)\varphi\|(\mathcal{T}x^* - \mathcal{T}^2x^*)\| + \alpha_n\varphi\|(\mathcal{T}x^* - \mathcal{T}^2x^*)\| + \delta\alpha_n(1-\beta_n)\varphi\|x^* - \mathcal{T}x^*\|$$
$$+ \delta\alpha_n\beta_n\varphi\|(\mathcal{T}x^* - \mathcal{T}^2x^*)\| + \{\delta(1-\alpha_n) + \delta\beta_n\}\|\mathcal{T}x_n - x^*\| + \delta(1-\beta_n)||x_n - x^*||$$

Since, $\varphi(0) = 0$, therefore,

$$\|y_n - x^*\| \ \leq \delta(1-\alpha_n+\beta_n)\|\mathcal{T}x_n - x^*\| + \delta(1-\beta_n)\|x_n - x^*\| \qquad (2.3)$$

Now, consider

$$\|\mathcal{T}x_n - x^*\| \leq \varphi\|x_n - \mathcal{T}x_n\| + \delta\|x_n - x^*\|$$
$$\leq \varphi\|x^* - \mathcal{T}x^*\| + \delta\|x_n - x^*\|$$

$$\leq \delta\|x_n - x^*\| \tag{2.4}$$

Hence, eqn. (2.3) becomes

$$\|y_n - x^*\| \ \leq \delta^2(1-\alpha_n+\beta_n)\|x_n - x^*\| + \delta(1-\beta_n)\|x_n - x^*\|$$

Further, eqn. (2.1) becomes

$$\|x_{n+1} - x^*\| \leq \varphi(\|x^* - \mathcal{T}x^*\|) + \delta^3(1-\alpha_n+\beta_n)\|x_n - x^*\| + \delta^2(1-\beta_n)\|x_n - x^*\|$$

Again, $\varphi(\|x^* - \mathcal{T}x^*\|) = 0$, hence,

$$\|x_{n+1} - x^*\| \leq \{\delta^3(1-\alpha_n+\beta_n) + \delta^2(1-\beta_n)\}\|x_n - x^*\|$$

$$\leq \delta^3(1-\alpha_n+\beta_n)\|x_n - x^*\| \tag{2.5}$$

Continuing the above process, the following estimates are obtained,

$$\|x_{n+1} - x^*\| \leq \prod_{r=1}^{n}[\delta^{3^r}(1-\alpha_r+\beta_r)] \ \ \|x_0 - x^*\| \tag{2.6}$$

As, $\delta \in (0,1)$

So, it is concluded from (2.6) that as $n \to \infty$, $\|x_{n+1} - x^*\| = 0$

Which suggests that the sequence $\{x_n\}$ is convergent to the fixed point x^* of the mapping $\mathcal{T}$.

Example 2.2 Let us assume a mapping defined as $\mathcal{T}: [0,1] \longrightarrow [0,1]$ given by $\mathcal{T}(x) = \frac{x}{2}$ such that $\mathcal{T}$ has a unique fixed point at $x = 0$. Also, let $\alpha_n = \frac{1}{2}$ and $\beta_n = \frac{2}{3}$ for all the values of n. Let $\varphi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be an increasing function and also tale $\delta = \frac{2}{3}$ then the sequence $\{x_n\}$ defined as in definition 1.3 is convergent to the fixed point of $\mathcal{T}$.

Solution: We first show that $\mathcal{T}$ satisfies the generalized f-contraction defined by definition 1.4. Consider,

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \varphi(\|x - \mathcal{T}x\|) + \delta\|x - y\|$$

where $x \ and \ y \in [0,1]$

Then, by definition and the values taken we have,

$$\left\|\frac{x}{2} - \frac{y}{2}\right\| \leq \varphi\left(\left\|x - \frac{x}{2}\right\|\right) + \frac{1}{2}\|x - y\|$$

$$\frac{1}{2}\|x - y\| \leq \varphi\left(\left\|\frac{x}{2}\right\|\right) + \frac{1}{2}\|x - y\| \tag{2.7}$$

As φ is an increasing function so the equation (2.5) holds for the mapping $\mathcal{T}$

Now, consider

$$\|x_{n+1} - x^*\| = \|\mathcal{T}y_n - 0\| = \frac{y_n}{2} \tag{2.8}$$

Now, $\|y_n - 0\| = \left\|\mathcal{T}\left(\frac{1}{2}\right)\mathcal{T}x_n + \frac{1}{2}\mathcal{T}z_n - 0\right\| = \left\|\frac{1}{2}\mathcal{T}\left(\frac{x_n}{2}\right) + \frac{1}{2}\frac{z_n}{2} - 0\right\|$

$$= \frac{1}{4} \|\mathcal{T}(x_n) + z_n\|$$

$$= \frac{1}{4} \left\| \frac{x_n}{2} + z_n \right\| \tag{2.9}$$

Also consider,

$$\|z_n - 0\| = \left\| \frac{x_n}{3} + \frac{2\mathcal{T}x_n}{3} \right\| = \frac{2}{3} x_n \tag{2.10}$$

From, (2.6) $y_n = \frac{7}{24} x_n$

So, from (2.5) $x_{n+1} = \frac{7}{48} x_n$

Which further states that $x_{n+1} = (\frac{7}{48})^n x_0$

And as $n \longrightarrow \infty, x_{n+1} \longrightarrow 0$

which shows that the sequence $\{x_n\}$ is convergent to the fixed point of $\mathcal{T}$

Now, we prove a similar result for another three-step iterative scheme which is Picard-Ishikawa hybrid iterative process [31].

Theorem 2.3: Let $\mathcal{D}$ represents a convex subset of a complex valued Banach space (E, ‖.‖). Let $\mathcal{T}: \mathcal{D} \rightarrow \mathcal{D}$ be a mapping which satisfies the condition (1.4). Let an iterative process be defined by a sequence $\{u_n\}$ in $\mathcal{D}$ as follows:

$$u_{n+1} = \mathcal{T}v_n$$

$$v_n = (1 - \alpha_n)u_n + \alpha_n \mathcal{T}w_n$$

$$w_n = (1 - \beta_n)u_n + \beta_n \mathcal{T}u_n$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in (0, 1). Then, the sequence $\{x_n\}$ is convergent to the fixed point of $\mathcal{T}$.

Proof: Consider that $\mathcal{T}$has a fixed point and let that point be given by u^*. To show that the sequence $\{u_n\}$ converges to the fixed point x^* consider,

$$\|u_{n+1} - u^*\| = \|\mathcal{T}v_n - u^*\|$$

$$\leq \varphi(\|v_n - \mathcal{T}v_n\|) + \delta\|v_n - u^*\|$$

$$\leq \varphi(\|u^* - \mathcal{T}u^*\|) + \delta\|v_n - u^*\| \tag{2.11}$$

Now,

$$\|v_n - u^*\| = \|(1 - \alpha_n)u_n + \alpha_n \mathcal{T}w_n - u^*\|$$

$$\leq (1 - \alpha_n)\|u_n - u^*\| + \alpha_n\|\mathcal{T}w_n - u^*\|$$

$$\leq (1 - \alpha_n)\|u_n - u^*\| + \alpha_n\{\varphi(\|w_n - \mathcal{T}w_n\|) + \delta\|w_n - u^*\|\}$$

$$\leq (1 - \alpha_n)\|u_n - u^*\| + \alpha_n\{\varphi(\|u^* - \mathcal{T}u^*\|) + \delta\|w_n - u^*\|\} \tag{2.12}$$

Since, $\varphi(\|u^* - \mathcal{T}u^*\|) = 0$,

$$\|v_n - u^*\| \le (1-\alpha_n)\|u_n - u^*\| + \alpha_n\delta\|w_n - u^*\| \tag{2.13}$$

Further,

$$\begin{aligned}\|w_n - u^*\| &= \|(1-\beta_n)u_n + \beta_n\mathcal{T}u_n - u^*\| \\ &\le (1-\beta_n)\|u_n - u^*\| + \beta_n\|\mathcal{T}u_n - u^*\| \\ &\le (1-\beta_n)\|u_n - u^*\| + \beta_n\{\varphi(\|u_n - \mathcal{T}u_n\|) + \delta\|u_n - u^*\|\}\end{aligned}$$

$$\le (1-\beta_n+\beta_n\delta)\|u_n - u^*\| + \beta_n\varphi(\|u^* - \mathcal{T}u^*\|) \tag{2.14}$$

Since, $\varphi(\|u^* - \mathcal{T}u^*\|) = 0$,

So,

$$\|w_n - u^*\| \le (1-\beta_n+\beta_n\delta)\|u_n - u^*\| \tag{2.15}$$

Hence from (2.13)

$$\|v_n - u^*\| \le (1-\alpha_n)\|u_n - u^*\| + \alpha_n\delta\{(1-\beta_n+\beta_n\delta)\|u_n - u^*\|\}$$

$$\|v_n - u^*\| \le \{(1-\alpha_n) + \alpha_n\delta(1-\beta_n+\beta_n\delta)\}\|u_n - u^*\| \tag{2.16}$$

Further, from (2.11)

$$\begin{aligned}\|u_{n+1} - u^*\| &\le \delta\{(1-\alpha_n) + \alpha_n\delta(1-\beta_n+\beta_n\delta)\}\|u_n - u^*\| \\ &\le \alpha_n\delta^2(1-\beta_n+\beta_n\delta)\|u_n - u^*\|\end{aligned}$$

$$\|u_{n+1} - u^*\| \le \prod_{i=1}^{n}[\alpha_i\delta^{2^i}(1-\beta_i+\beta_i\delta)]\|u_0 - u^*\| \tag{2.17}$$

Since $\delta \in (0,1)$

So, it is concluded from (2.17) that as $n \to \infty$, $\|u_{n+1} - u^*\| = 0$

Which suggests that the sequence $\{u_n\}$ converges to the fixed point u^*

Hence, the convergence of Picard Ishikawa hybrid iterative process is established.

3. Analysis of Convergence Rate of K-Iterative Scheme and Picard Ishikawa Hybrid Iterative Process

In this section, we take an example to establish the convergence of K-iterative process and Picard Ishikawa Hybrid process with and hence show that the K-iterative process is converging faster as compared to Picard Ishikawa Hybrid process.

Example 3.1: Consider a mapping $\mathcal{T}: [0,1] \longrightarrow [0,1]$ defined as $\mathcal{T}(x) = \sqrt{x+2}$ such that $\mathcal{T}$ has a unique fixed point at $x = 2$. Also, let $\alpha_n = \frac{1}{2}$ and $\beta_n = \frac{2}{3}$ for all the values of n. Let a sequence $\{x_n\}$ be defined as in definition (1.3) and another sequence $\{u_n\}$ be defined as in definition (1.4). Then, we have the following results for two different initial approximations:

Table 1 - Comparison of K-Iteration and Picard Ishikawa Hybrid Process {Assuming the initial values $x_0 = u_0 = 1.99$ $(Close\ to\ fixed\ point)$}

Step	K-Iteration Process	Picard Ishikawa Hybrid Process
1	2.029409080977518	1.998593173738072
2	2.0003438936691715	1.9998021536602233
3	2.0000040299018327	1.9999721776330923
4	2.0000000472253983	1.9999960874751967
5	2.0000000005534226	1.9999994498011113
6	2.0000000000064855	1.9999999226282794
7	2.000000000000076	1.9999999891196016
8	2.000000000000001	1.999999998469944
9	**2.0**	1.999999999784836
10	2.0	1.9999999999697424
11	2.0	1.999999999995745
12	2.0	1.9999999999994016
13	2.0	1.9999999999999158
14	2.0	1.999999999999988
15	2.0	1.9999999999999982
16	2.0	1.9999999999999998
17	2.0	**2.0**
18	2.0	2.0
19	2.0	2.0
20	2.0	2.0

Table 2 - Comparison of K-Iteration and Picard Ishikawa Hybrid Process {Assuming the initial values $x_0 = u_0 = 5.0$ $(Far\ from\ fixed\ point)$}

Step	K-Iteration Process	Picard Ishikawa Hybrid Process
1	2.029409080977518	2.3801621278686764
2	2.0003438936691715	2.0526543330554925
3	2.0000040299018327	2.007388624656347
4	2.0000000472253983	2.001038711222037
5	2.0000000005534226	2.000146062554106
6	2.0000000000064855	2.0000205399238373
7	2.000000000000076	2.0000028884243606
8	2.000000000000001	2.0000004061846277
9	**2.0**	2.0000000571197125
10	2.0	2.0000000080324596
11	2.0	2.0000000011295644
12	2.0	2.000000000158845
13	2.0	2.0000000000223377
14	2.0	2.000000000003141
15	2.0	2.0000000000004414
16	2.0	2.000000000000062
17	2.0	2.0000000000000084
18	2.0	2.000000000000001
19	2.0	**2.0**
20	2.0	2.0

Observation 3.2: From the above calculations, it has been analyzed that K-iterative scheme is converging faster to the fixed point of the mapping as compared to Picard Ishikawa Hybrid process. Also, the assumption of the initial value (close of far from fixed point) has negligible impact on the step achieving a fixed point.

4. Conclusion

From the above result it is hence concluded that the two different three-step iteration schemes considered in the above study: K iteration process and Picard Ishikawa Hybrid iterative process converge under the setting of generalized F- contractions for a complex valued Banach space. For the similar conditions and initial assumption, the K-iterative scheme tends to converge faster as compared to Picard Ishikawa Hybrid Process. This enhances the future scope of study in this area where different combinations of conditions and iterative schemes can be utilized for fixed point convergence and analysis of their rate of convergence can be also be studied.

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