www.revistageintec.net
ISSN: 2237-0722
revista GEINTEC
Gestão, Inovação e Tecnologias

Steady-state Analysis of an $M/M/2$ Queueing System Operating in a Multi-phase Random Environment Subject to Disaster and Repair

D. Piriadarshani[1]; S. Narasimhan[2]; Kanimozhiraman[3]

[1]Department of Mathematics, Hindustan Institute of Technology and Science, Chennai, India.
[1]piriadarshani@hindustanuniv.ac.in
[2] Department of Mathematics, Hindustan Institute of Technology and Science, Chennai, India.
Department of Mathematics, KCG College of Technology, Chennai, India.
[2]narasimhan@kcgcollege.com
[3]Department of Mathematics, KCG College of Technology, Chennai, India.
[3]kanimozhi@kcgcollege.com

Abstract

In this research article, an $M/M/2$ queueing system operating in a multi-phase random environment subject to disaster and repair is studied. The random environment discussed here has N phases and the k - th phase is exponentially distributed with mean $1/\eta_k$, $k = 1,2,..., N$. The queueing system behaves like $M(\lambda_k)/M(\mu_k)/2$ while in the k - th phase. At the end of the k - th phase, a disaster occurs where the servers are taken for repair and all the customers in the system are wiped out. Both the servers are repaired jointly and the repair time is exponentially distributed with mean $1/\eta_0$, Immediately after the repair completions, the system goes to phase k with probability q_k . During repair time, Customers are permitted to enter the queueing system. The probabilistic behaviour of the system is studied in steady state by using probability generating function technique. Some performance measures of this queueing model are also obtained.

Key-words: Disaster, Wash-Out, Repair, Random Environment, Steady State Analysis.
AMS Subject Classification: 60K20, 60K25, 90B22

1. Introduction

Several researchers have studied many-server queueing systems (see, for example Karlin and McGregor [4], Takacs [11], Kleinrock [5], Natarajan [9], and Dharmaraja and Rakesh Kumar [3]). A few researchers have obtained transient solution of many-server queueing systems (see Parthasarathy

Received: 20.04.2021 – Accepted: 12.05.2021

and Sharafali [10], Krishna Kumar and Arivudainambi [6], Krishna Kumar and Madheswari [7] and - Seedy et al. [1],). On the other hand, many authors have paid their attention to study some special class of queueing systems subject to impatience of customers and/or randomly occurring disasters (see for example, Sengupta [12], Chakravarthy [2], Krishna Kumar et al. [8]). The steady-state behavior of an $M/M/1$ queue operating in random environment subject to disasters where the underlying environment is described by a n phase continuous-time Markov chain have been analyzed. The time-dependent behaviour of the above said model has also been analyzed. However, to the best of our knowledge, $M/M/2$ queue systems operating in a random environment subject to disasters and repair have not been studied so far in literature. In this research paper, we block this gap by obtaining a steady-state analysis of a M/M/2 queueing system operating in a uncertain random environment subject to disasters and repairs. The sections in this paper are arranged in the following manner: Section 2 briefly describes the model. In section 3, the time-dependent probabilities of the system are discussed and its the integral equations are derived. In section 4, explicit expressions of the steady-state probabilities of the above mentioned queueing system are obtained.

2. Model Description

Let us consider a two server queueing system working in a random environment. We assume that the environment is in any one of the $m+1$ states 0, 1, 2, $\ldots, N$. The environmental state 0 represents the state that the servers are jointly undergoing repair. The random repair time is an exponential random variable distributed with mean $\frac{1}{\eta_0}$, In the course of repair time, customers join the system according to Poisson process with arrival rate λ_0. Immediately after the repair, the system goes to phase $k, k=1,2,\ldots,N$ with probability q_k; such that $\sum_{k=1}^{N} q_k = 1$. The system resides in phase k for a random interval of time that is exponentially distributed with mean $1/\eta_k$ and at the end of the residing period, all customers in the system are washed out and the system moves to phase 0: When the environment is in phase k, the system behaves like an $M(\lambda_k)/M(\mu_k)/2$ queue with arrival rate λ_k and service rate μ_k.

During time $t=0$, we presume that a catastrophe has just occurred so that the system is in phase 0 (repair state). Let $E(t)$ denote the phase of the environment at time t and at time t, let $X(t)$ denotes the number of customers in the queueing system. Then the joint process $\{(X(t),E(t))|t\geq 0\}$ is a continuous time parameter Markov process whose state space is

$$\{(j,k)|j=0,1,2,\cdots;k=0,1,2,\cdots,N\}$$

We define the state probability as follows:

$$\{p(j,k,t)=\Pr[X(t)=j,E(t)=k|X(0)=0,E(0)=0]k=0,1,2,\cdots N;\ j=0,1,2,\cdots\} \quad (1)$$

In the next section, we derive the integral equations for $p(j,k,t)$

3. To Obtain the Integral Equations in Steady-State

Using the convolution operation

$$f(t)\Theta g(t)=\int_0^t f(u)g(t-u)du \quad (2)$$

and renewal theoretic arguments, we find the following integral equations:

Case 1: $j=0,k=0$

$$p(0,0,t)=e^{-(\lambda_0+\eta_0)t}+\sum_{k=1}^{N}\sum_{j=0}^{\infty}p(j,k,t)\eta k\,\Theta\, e^{-(\lambda_0+\eta_0)t} \quad (3)$$

Case 2: $j=1,2,...,k=0$

$$p(j,0,t)=p(j-1,0,t)\lambda_0 © e^{-(\lambda_0+\eta_0)t} \quad (4)$$

Case 3: $j=0,k=1,2,...,N$

$$p(0,k,t)=p(0,0,t)\eta_0 q_k © \ e^{-(\lambda_k+\eta_k)t}+\ p(1,k,t)\mu_k\ © \ e^{-(\lambda_k+\eta_k)t} \quad (5)$$

Case 4: $j=1,k=1,2,...,N$

$$p(1,k,t)=p(1,0,t)\eta_0 q_k © \ e^{-(\lambda_k+\mu_k+\eta_k)t}+\ p(2,k,t)2\mu_k\ © \ e^{-(\lambda_k+\mu_k+\eta_k)t} \quad (6)$$

Case 5: $j=2,3,...,k=1,2,...,N$

$$p(j,k,t)=p(j,0,t)\eta_0 q_k © \ e^{-(\lambda_k+2\mu_k+\eta_k)t}+\ p(j-1,k,t)\lambda_k\ ©$$

$$e^{-(\lambda_k+2\mu_k+\eta_k)t}+\ p(j+1,k,t)2\mu_k\ © \ e^{-(\lambda_k+\mu_k+\eta_k)t} \quad (7)$$

ISSN: 2237-0722
Vol. 11 No. 3 (2021)
Received: 20.04.2021 – Accepted: 12.05.2021

4. Steady-state Solution

The limiting state probabilities are defined by

$$\pi(j,k)=\lim_{t\to\infty} p(j,k,t) \tag{8}$$

Using the Laplace transform's final value theorem, equation (8) gives

$$\pi(j,k)=\lim_{s\to 0} sp^{*}(j,k,s), \tag{9}$$

Where $p*(j,k,t)$ is the Laplace transform of $p(j,k,t)$ Applying Laplace transform on both sides of equations (3)-(7), we obtain

$$(s+\lambda_0+\eta_0)p*(0,0,s)=1+\sum_{k=1}^{N}\sum_{j=0}^{\infty} p*(j,k,t)\eta_k, \tag{10}$$

$$(s+\lambda_0+\eta_0)p*(j,0,s)=p*(j-1,0,s)\lambda_0,\ j=1,2,... \tag{11}$$

$$(s+\lambda_k+\eta_k)p*(0,k,s)=p*(0,0,s)\eta_0 q_k+p*(1,k,s)\mu_k, k=1,2,...,N; \tag{12}$$

$$(s+\lambda_k+\mu_K+\eta_k)p*(1,k,s)=p*(1,0,s)\eta_0 q_k+ \\ p*(0,k,s)\lambda_k+p*(2,k,s)2\mu_k, k=1,2,...,N; \tag{13}$$

$$(s+\lambda_k+2\mu_K+\eta_k)p*(j,k,s)=p*(j,0,s)\eta_0 q_k+ \\ p*(j-1,k,s)\lambda_k+p*(j+1,k,s)2\mu_k,\ j=2,3,...:k=1,2,...,N. \tag{14}$$

Multiplying both sides of equations (10)-(14) with s and applying equation (9), we obtain

$$(\lambda_0+\eta_0)\pi(0,0)=1+\sum_{k=1}^{N}\sum_{j=0}^{\infty}\pi(j,k)\eta_k, \tag{15}$$

$$(\lambda_0+\eta_0)\pi(j,0)=\pi(j-1,0)\lambda_0,\ j=1,2,...; \tag{16}$$

$$(\lambda_k+\eta_k)\pi(0,k)=\pi(0,0)\eta_0 q_k+\pi(1,k)\mu_k, k=1,2,...,N \tag{17}$$

$$(\lambda_k+\mu_k+\eta_k)\pi(1,k)=\pi(1,0)\eta_0 q_k+\pi(0,k)\lambda_k+\pi(2,k)2\mu_k, k=1,2,...,N \tag{18}$$

$$(\lambda_k+2\mu_k+\eta_k)\pi(j,k)=\pi(j,0)\eta_0 q_k+\pi(j-1,k)\lambda_k+\pi(j+1,k)2\mu_k, \\ j=2,3,...;k=1,2,...,N \tag{19}$$

ISSN: 2237-0722
Vol. 11 No. 3 (2021)
Received: 20.04.2021 – Accepted: 12.05.2021

The system of equations (16)-(19) together with the total probability law can be explicitly solved. The result is given in the following theorem:

4.1Theorem. The limiting state probabilities are given by

$$\pi(0,0)=\frac{1}{(\lambda_0+\eta_0)\left[\frac{1}{\eta_0}+\sum_{k=1}^{N}\frac{q_k}{\eta_k}\right]};$$

$$\pi(j,0)=\left(\frac{\lambda_0}{\lambda_0+\eta_0}\right)^j\pi(0,0),\ j=1,2,...;$$

$$\pi(0,k)=\frac{\eta_0 q_k\beta_k\left[\frac{(\lambda_0+\eta_0)}{(\lambda_0+\eta_0-\lambda_0\beta_k)}+(1-\beta_k)\right]}{(1-\beta_k)[(\lambda_k+\eta_k)\beta_k+2\mu_k]}\pi(0,0),\ k=1,2,..N;$$

$$\pi(j,k)=\eta_0 q_k\left\{\sum_{i=0}^{j}\sum_{r=0}^{\infty}\pi(j-1+r,0)\frac{\lambda_k{}^{l}\beta_k^{l+r+1}}{2^{l+1}\mu_k^{l+1}}\right\}$$

$$+\pi(0,k)\left[(\lambda_k+\eta_k)\frac{\lambda_k{}^{j-1}\beta_k^{j}}{2^j\mu_k^j}+\{2\mu_k-(\lambda_k+\eta_k)(1-\beta_k)\}\frac{\lambda_k{}^{j}\beta_k^{j+1}}{2^{j+1}\mu_k^{j+1}}\right]$$

$$+\pi(0,0)\eta_0 q_k\left[(1+\beta_k)\frac{\lambda_k{}^{j}\beta_k^{j+1}}{2^{j+1}\mu_k^{j+1}}-\frac{\lambda_k{}^{l-1}\beta_k^{j}}{2^j\mu_k^j}\right],\ j=1,2,...;k=1,2,...,N,$$

Where $\beta_k=\frac{(\lambda_k+2\mu_k+\eta_k)-\sqrt{(\lambda_k-2\mu_k+\eta_k)^2+8\mu_k\eta_k}}{2\lambda_k},k=1,2,...,N.$

Proof:

From equation (16), we get $\pi(j,0)=\left(\frac{\lambda_0}{\lambda_0+\eta_0}\right)^j\pi(0,0),\ j=1,2,...;$ (20)

define $G_k(\theta)=\sum_{j=0}^{\infty}\pi(j,k)\theta^j,k=0,1,2,...,N.$

From equation (20), we get $G_0(\theta)=\frac{(\lambda_0+\eta_0)}{(\lambda_0+\eta_0-\lambda_0\theta)}\pi(0,0),$ (21)

Putting $\theta=1$ in equation (21), we get $G_0(1)=\frac{(\lambda_0+\eta_0)}{\eta_0}\pi(0,0),$ (22)

We Multiply equation (19) by θ^j on both sides and summing from $\theta=2$ to ∞, we get

$$(\lambda_k + 2\mu_k + \eta_k)\sum_{j=2}^{\infty}\pi(j,k)\theta^j = \eta_0 q_k \sum_{j=2}^{\infty}\pi(j,0)\theta^j + \lambda_k \sum_{j=2}^{\infty}\pi(j-1,k)\theta^j + 2\mu_k \sum_{j=2}^{\infty}\pi(j+1,k)\theta^j \quad (23)$$

Simplifying equation (23) by using equations (17) and (18), we get

$$G_k(\theta) = \frac{\eta_0 q_k \theta[G_0(\theta) + \pi(0,0)(1-\theta)] - \pi(0,k)(1-\theta)[(\lambda_k + \eta_k)\theta + 2\mu_k]}{(\lambda_k + 2\mu_k + \eta_k)\theta - \lambda_k\theta^2 - 2\mu_k}, k = 1,2,...N. \quad (24)$$

Putting $\theta = 1$ in (24) and using (22), we get

$$G_k(1) = \frac{(\lambda_0 + \eta_0)q_k \pi(0,0)}{\eta_k}, k = 1,2,.... \quad (25)$$

By total probability law, we get $\sum_{k=0}^{N}\sum_{j=0}^{\infty}\pi(j,k) = 1.$ (26)

From equation (26), we get n $\sum_{k=0}^{N} G_k(1) = 1$ (27)

Substituting equations (22) and (25) in equation (27), we obtain

$$\pi(0,0) = \frac{1}{(\lambda_0 + \eta_0)\left[\frac{1}{\eta_0} + \sum_{k=1}^{N}\frac{q_k}{\eta_k}\right]} \quad (28)$$

It is evident that $G_k(\theta)$ is analytic inside the unit disk $|\theta| < 1$. Consequently, the numerator of the right hand side of (24) disappears at the zeros of the denominator of the right hand side of (24).

The zeros of the denominator of the right hand side of (24) are given by

$$(\lambda_k + 2\mu_k + \eta_k)\theta - \lambda_k\theta^2 - 2\mu_k = 0. \quad (29)$$

The discriminant of the quadratic equation (29) is

$$(\lambda_k + 2\mu_k + \eta_k)^2 - 8\lambda_k\mu_k = (\lambda_k + 2\mu_k + \eta_k)^2 + 8\lambda_k\mu_k.$$

Hence both the roots of equation (29) are real, and they are given by

$$\alpha_k = \frac{(\lambda_k + 2\mu_k + \eta_k) + \sqrt{(\lambda_k + 2\mu_k + \eta_k)^2 + 8\mu_k\mu_k}}{2\lambda_k}. \quad (30)$$

$$\beta_k = \frac{(\lambda_k + 2\mu_k + \eta_k) - \sqrt{(\lambda_k + 2\mu_k + \eta_k)^2 + 8\mu_k\mu_k}}{2\lambda_k} \quad (31)$$

The product of the roots of equation (29) is $\alpha_k\beta_k = \frac{2\mu_k}{\lambda_k}$. For stable solution, we assume that

$2\mu_k > \lambda_k$. Then $2\mu_k + \eta_k > 2\mu_k > \lambda_k$. It is easy to establish that $0 < \beta_k < 1$ Further, we find that $\alpha_k > 1$ Using the roots α_k and β_k equation (24) leads to

$$G_k(\theta) = \frac{\eta_0 q_k \theta [G_0(\theta) + \pi(0,0)(1-\theta)] - \pi(0,k)(1-\theta)[(\lambda_k + \eta_k)\theta + 2\mu_k]}{\lambda_k(\alpha_k - \theta)(\theta - \beta_k)}, k = 1,2,...N \quad (32)$$

Invoking the analyticity of $G_k(\theta)$ in $|\theta| < 1$, the numerator on the right hand side of equation

(32) vanishes at $\theta = \beta_k$ Consequently, we get

$$\eta_0 q_k \beta_k [G_0(\beta_k) + \pi(0,0)(1-\beta_k)] - \pi(0,k)(1-\beta_k)[(\lambda_k + \eta_k)\beta_k + 2\mu_k] = 0 \quad (33)$$

Using equation (33), we get

$$\pi(0,k) = \frac{\eta_0 q_k \beta_k \left[\frac{(\lambda_0 + \eta_0)}{(\lambda_0 + \eta_0 - \lambda_0 \beta_k)} + (1-\beta_k) \right]}{(1-\beta_k)[(\lambda_k + \eta_k)\beta_k + 2\mu_k]} \pi(0,0), k = 1,2,..., N \quad (34)$$

Using equation (33) in equation (32), and after much algebraic simplification, we obtain

$$G_k(\theta) = \eta_0 q_k \sum_{j=0}^{\infty} \left\{ \sum_{i=0}^{j} \sum_{r=0}^{\infty} \pi(j-1+r,0) \frac{\lambda_k{}^l \beta_k^{l+r+1}}{2^{l+1} \mu_k^{l+1}} \right\} \theta^j$$

$$+ \pi(0,k) \left[(\lambda_k + \eta_k) \sum_{j=1}^{\infty} \frac{\lambda_k{}^{j-1} \beta_k^j \theta^j}{2^j \mu_k^j} + \{2\mu_k - (\lambda_k + \eta_k)(1-\beta_k)\} \sum_{j=0}^{\infty} \frac{\lambda_k{}^j \beta_k^{j+1} \theta^j}{2^{j+1} \mu_k^{j+1}} \right]$$

$$+ \pi(0,0) \eta_0 q_k \left[(1+\beta_k) \sum_{j=0}^{\infty} \frac{\lambda_k{}^j \beta_k^{j+1} \theta^j}{2^{j+1} \mu_k^{j+1}} - \sum_{j=1}^{\infty} \frac{\lambda_k{}^{l-1} \beta_k^j \theta^j}{2^j \mu_k^j} \right], k = 1,2,..., N \quad (35)$$

Equating the coeffcients of θ^0 on both sides of equation (35), we get back equation (34).

Equating the coeffcients of θ^j, $j = 1,2,...$, on both sides of equation (35), we obtain

$$\pi(j,k) = \eta_0 q_k \left\{ \sum_{i=0}^{j} \sum_{r=0}^{\infty} \pi(j-1+r,0) \frac{\lambda_k{}^l \beta_k^{l+r+1}}{2^{l+1} \mu_k^{l+1}} \right\}$$

$$+ \pi(0,k) \left[(\lambda_k + \eta_k) \frac{\lambda_k{}^{j-1} \beta_k^j}{2^j \mu_k^j} + \{2\mu_k - (\lambda_k + \eta_k)(1-\beta_k)\} \frac{\lambda_k{}^j \beta_k^{j+1}}{2^{j+1} \mu_k^{j+1}} \right]$$

$$+ \pi(0,0) \eta_0 q_k \left[(1+\beta_k) \frac{\lambda_k{}^j \beta_k^{j+1}}{2^{j+1} \mu_k^{j+1}} - \frac{\lambda_k{}^{l-1} \beta_k^j}{2^j \mu_k^j} \right], j = 1,2,...; k = 1,2,..., N \quad (36)$$

The proof is now complete.

5. Performance Measures

To illustrate, we assume the below mentioned values for the parameters of our model subject to the stability condition $2\mu_k > \lambda_k, k = 1,2,..., N$:

$$\eta_0 = 1.5; \lambda_0 = 1.2;$$

$q_1 = 0.2$	$\eta_1 = 0.03$	$\lambda_1 = 1.8$	$\mu_1 = 2.6$
$q_2 = 0.3$	$\eta_2 = 0.04$	$\lambda_1 = 1.6$	$\mu_1 = 2.4$
$q_3 = 0.2$	$\eta_3 = 0.01$	$\lambda_1 = 1.5$	$\mu_1 = 2.5$
$q_4 = 0.2$	$\eta_4 = 0.02$	$\lambda_1 = 1.7$	$\mu_1 = 2.7$
$q_5 = 0.1$	$\eta_5 = 0.03$	$\lambda_1 = 1.4$	$\mu_1 = 2.4$

We have computed the steady-state probabilities $\pi(j,k)$ and obtained the following table:

Table 1: Steady-state probabilities

j	$\pi(j,0)$	$\pi(j,1)$	$\pi(j,2)$	$\pi(j,3)$	$\pi(j,4)$	$\pi(j,5)$
0	0.0077	0.0672	0.0779	0.2235	0.1082	0.0379
1	0.0034	0.0465	0.0518	0.1341	0.068	0.0221
2	0.0015	0.0161	0.0173	0.0403	0.0215	0.0065
3	0.0007	0.0056	0.0058	0.0121	0.0068	0.0019
4	0.0003	0.0019	0.002	0.0037	0.0022	0.0006
5	0.0001	0.0007	0.0007	0.0011	0.0007	0.0002
6	0.0001	0.0002	0.0002	0.0003	0.0002	0.0001
7	0	0.0001	0.0001	0.0001	0.0001	0
8	0	0	0	0	0	0
9	0	0	0	0	0	0
10	0	0	0	0	0	0

5.1 Mean Queue Size in *k*-th phase, *k* = 0,1,2,…, N

Let $E[L_k]$ denote the stationary mean number of customers in the k-phase. Then, we have

$$E[L_k] = \sum_{j=1}^{\infty} j\pi(j,k), k = 0,1,2,..., N. \qquad (37)$$

Then, the stationary mean total number of customers in the queueing system is obtained by

$$E[L] = \sum_{j=0}^{\infty}\sum_{k=0}^{N} j\pi(j,k) = \sum_{k=0}^{N}\sum_{j=0}^{\infty} j\pi(j,k) = \sum_{k=0}^{N} E[L_k], k = 0,1,2,..., N. \qquad (38)$$

5.2 Mean Number of Customers Washed Out by Disasters per Unit Time from the System

If C denote the number of customers wiped out from the queueing system per unit time, then

$$E[C] = \sum_{k=1}^{N} \eta_k \sum_{j=1}^{\infty} j\pi(j,k) = \sum_{k=1}^{N} \eta_k E[L_k]. \quad (39)$$

Using table 1, we obtain the following table of mean values:

$E[L_0] = 0.0111, E[L_1] = 0.1091, E[L_2] = 0.1173, E[L_3] = 0.2743$

$E[L_4] = 0.1455, E[L_5] = 0.0444, E[L] = 0.7017, E[C] = 0.0149$

6. Conclusion

Steady -state analysis of an $M/M/2$ queueing system is discussed which was operating on a multi-phase uncertain random environment subject to disaster and repair and its steady state probability is obtained. Further, we also seek the characteristics of $M/M/2$ queueing system which operates in a multi-phase uncertain environment subject to repair and disaster .

References

R.O. Al-Seedy, A.A. El-Sherbiny S.A. El-Shehawy S. I. Ammar. Transient solution of the M/M/c queue with balking and reneging, *Computers and Mathematics with Applications,* 57. 1280-1285, (2009).

S.R. Chakravarthy, "A disaster queue with Markovian arrivals and impatient customers," *Applied Mathematics and Computation, 214,* 48-59, (2009).

S. Dharmaraja and Rakesh Kumar, *Transient solution of a Markovian queuing model with heterogeneous servers and catastrophes,* OPSEARCH, vol. 52, 810-826, (2015).

S. Karlin and J. McGregor. Many server queueing processes with Poisson input and exponential service times, *Pacific Journal of Mathematics, 8,* 87-118, (1958).

L. Kleinrock, *Queueing Systems:* Volume I – Theory. New York: Wiley Interscience, 1975.

B. Krishna Kumar and D. Arivudainambi, *Transient solution of an M/M/c queue with heterogeneous servers and balking,* Int. J. Inf. Manag. Sci., vol. 12, 15-27, (2001).

B. Krishna Kumar, S. Pavai Madheswari, *Transient behaviour of the M/M/2 queue with catastrophes, Statistica,* vol. LXII, 129-136, (2002).

B. Krishna Kumar, A. Vijayakumar, and S. Sophia, "Transient analysis of a Markovian queue with chain sequence rates and total catastrophes," *International Journal of Operational Research, 5,* 375-391. (2009).

G. Natarajan, *Analysis of Queues: Methods and Application,* CRC Press, 2012.

P.R. Parthasarathy and M. Sharafali, Transient solution to many server Poisson queue: A simple approach, *Journal of Applied Probability, 26,* 584-594, (1986).

L. Takacs, *Introduction to the theory of queues*, Oxford University Press, New York, 1962.

B. Sengupta, "A Queue with Service Interruptions in an alternating random environment," *Operations Research, 38,* 308-318. (1990).