www.revistageintec.net
ISSN: 2237-0722
revista GEINTEC
Gestão, Inovação e Tecnologias

Some Properties of the Covariant Functor Set of Exponential Type

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Abstract

In this paper, it is shown that the sets of all non-empty subsets $\mathrm{Set}(X)$ *of a topological space* X *with exponential topology is a covariant functor in the category of* Top*-topological spaces and their continuous mappings into itself.*

It is shown that the functor Set *is a covariant functor in the category of topological spaces and continuous mappings into itself, a pseudometric in the space* $\mathrm{Set}(X)$ *is defined, and compact, connected, finite, and countable subspaces of* $\mathrm{Set}(X)$ *are distinguished. It also shows various kinds of connectivity, soft, locally soft, and* $n-$*soft mappings in* $\mathrm{Set}(X)$. *One interesting example is given for the* TOP_Y *category. It is proved that the functor* Set *maps open mappings to open, contractible and locally contractible spaces and into contractible and locally contractible spaces. Next, we study the problem of the propagation of mappings in the space* $\mathrm{Set}(X)$ *and distinguish which sets the basic open sets of the space* $\mathrm{Set}(X)$ *consist of. The following takes place:*

a) The Set *functor is a covariant functor in the* Top *category;*

b) The functor $\mathrm{Set}: Top \to Top$ *preserves the layers of a continuous mapping, that is,* $(\mathrm{Set}(f))^{-1}(f(A)) = \mathrm{Set}(f^{-1}(A))$.

c) The functor Set *preserves the contractibility of topological spaces.*

Key-words: Functors, Mappings, Connection, Soft Mappings, Contractility, Homotopy, Absolute Extensors.

1. Introduction

Let X be a topological space. Let $\mathrm{Set}(X)$ denote the family of all nonempty subsets of it, that is, $\mathrm{Set}(X)=\{F:F\subset X,F\neq\varnothing\}$. Let $\gamma=\{F_1,F_2,F_3,...,F_s\}$ be a finite family of nonempty open subsets of the space X, that is, $\gamma=\{F_1,F_2,F_3,...,F_s\}$, where $F_i\neq 0$ and F_i is open in X. Consider a set of the form:

$$O_\gamma(F_1,F_2,...,F_s)=\{A:A\subset\bigcup_{i=1}^{s}F_i \text{ and } A\bigcap F_i\neq\varnothing \text{ for } i=\overline{1,s}\}.$$

The families of all sets of this kind are $O(F_1,F_2,...,F_s)$, i.e. $B\equiv\{O(F_1,F_2,...,F_s):F_i-$ is open in X and non-empty form, by definition, the base of the exponential topology Vietoris of the $\mathrm{Set}\ X$.

If X is a metric space with metric ρ, then, following Hausdorff, it is natural to introduce the pseudometrics ρ'_H on the family $\mathrm{Set}(X)$ using the formula:

$$\rho'_H(A,B)=\max\{\sup_{a\in A}\rho(a,B),\sup_{b\in B}\rho(b,A)\},$$

those. a topological space with this ρ'_H Hausdorff pseudometric is a $(\mathrm{Set}(X),\rho'_H)$ pseudo-metric space.

Recall that a mapping $\rho(x,y):X\times X\to R_+$ is called pseudometric on a set X if $\rho(x,y)$ satisfies the following conditions:

P1. $\rho(x,x)=0$ for any X;

P2. $\rho(x,y)=\rho(y,x)$ for any $x\in X$, $y\in X$;

P3. $\rho(x,y)+\rho(y,z)\geq\rho(x,z)$ for any $x,y,z\in X$.

If we consider the family $\exp X=\{F:F\subseteq X,F-$ is closed in $X\}$ of all non-empty closed subsets of the space X. This ρ'_H Hausdorff pseudo-metric will be the metric on $expX$. Obviously, $\exp X$ is a subspace of $\mathrm{Set}(X)$.

Consider a mapping $i:X\to\mathrm{Set}(X)$ assigning to each point x a set $i(X)$ consisting of exactly this one point x, i.e. $i(x)=\{x\}\in\mathrm{Set}(X)$. For this, the mapping i satisfies the equality

$$i^{-1}(O(F_1,F_2,...,F_k))=\bigcap_{i=1}^{k}F_i. \qquad (1)$$

ISSN: 2237-0722
Vol. 11 No. 2 (2021)
Received: 10.03.2021 – Accepted: 12.04.2021

This means that the mapping i is continuous. For a family γ consisting of one set Γ, we obtain from this that $i(\Gamma)=O(\Gamma)\bigcap X$. Hence, $i-$open mapping. Since it is one-to-one, by virtue of equality (1) it will be a topological embedding of the space X into the space $\mathrm{Set}(X)$. Hence, for any topological space X , we can assume that X - topologically lies in $\mathrm{Set}(X)$.

Thus, for any topological space X, a topological space $\mathrm{Set}(X)$ is defined, which topologically contains the given space X as a subset.

Each mapping $f:X\to Y$ can be considered as follows $f:X\to Y\to \mathrm{Set}Y$. Hence, each mapping $f:X\to \mathrm{Set}(Y)$ will be denoted by $\mathrm{Set}(f):\mathrm{Set}(X)\to \mathrm{Set}(Y)$ or, in short, by the $\mathrm{Set}(f)$-mapping defined by the formula:

$(\mathrm{Set}\ f)(A)=\cup\{f(a):a\in A\}$ for each $A\in \mathrm{Set}(X)$.

The following takes place

Lemma 1 [1]. If $f:X\to Y$ is a continuous mapping between topological spaces X and Y , then $\mathrm{Set}(f):\mathrm{Set}(X)\to \mathrm{Set}(Y)$ is a continuous mapping between $\mathrm{Set}(X)$ and $\mathrm{Set}(Y)$.

In this case, for any continuous mapping $f:X\to Y$, the following diagram holds

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow i_X & & \downarrow i_Y \\ \mathrm{Set}(X) & \xrightarrow{\mathrm{Set}(f)} & \mathrm{Set}(Y) \end{array} \qquad (2)$$

Let X be a topological space. We denote by $\exp X=\{F:\overline{F}=F,F\subseteq X,F\neq\varnothing\}$ the family of all nonempty closed subsets of it;

$\exp^{c}(X)=\{F:F-$ is compact; $F\subseteq X\}$ is the family of all compact subsets;

$\exp^{con}(X)=\{F:F-$ is connected, $F\subseteq X\}$ is the family of all connected subsets;

$\exp^{ccon}(X)=\{F:F-$ is connected and $F-$ is compact $F\subseteq X$ $\}$ is the family of all connected compact subsets of X .

Each of these families will be considered by the aforementioned Vystoris exponential topology.

Note that for any space X each of these topological spaces is a subspace of $Set(X)$. We obtain the following chain of spaces:

$$\exp^{ccon}(X) \subseteq \exp^{con}(X) \subseteq \exp(X) \subseteq Set(X) \tag{3}$$

$$\exp^{c}(X) \subseteq \exp X \tag{4}$$

Definition [2]. A non-empty space X is called $k-$connected $(0 \le k \le \infty)$ (denoted by $X \in C^r$) if every continuous mapping $f : S^r \to X$ with $r \le k$ is homotopic to a constant mapping $f : S^r \to \{x_0\}$ where $x_0 \in X$.

This definition is equivalent to the following:

If any mapping $f : S^r \to X$ can be extended to a continuous mapping $f : D^{r+1} \to X$, where $BdD^{r+1} = S^r$, D^{r+1} is closed, $BdD^{r+1} -$ denotes the boundary D^{r+1}.

Recall that a space X is called locally $k-$connected (denoted by $X \in LC^n$) if it is locally $k-$connected for $k = 0,1,...,n$, i.e. if, for any point $y_0 \in X$ and any of its neighborhoods V, there exists another neighborhood V_0 of it contained in V and such that any mapping $f : S^k \to V_0$, where $k \le n$, is homotopic to a constant mapping $f : S^k \to V$.

It is easy to see that a (local) $0-$connection is simply a (Local) path connection. For $k = 1$, our definition is equivalent to the standard definition of (local) simply connected.

We also recall that $X \in C$ (respectively, $X \in LC$) means the contractibility (respectively, local contractibility) of the space X.

a. $C \Rightarrow C^{\infty} \Rightarrow C^n$, where $n = 0,1,2,...$
b. $LC \Rightarrow LC^{\infty} \Rightarrow LC^n$, where $n = 0,1,2,...$
c. $C^n \Rightarrow C^m$ for $n \ge m$.
d. $LC^n \Rightarrow LC^m$ for $n \ge m$.

Recall that a space X is called linear-connected if for any two points $x_0, x_1 \in X$ there is a path connecting them in X. A space X is called contractible if the identity mapping $id_X : X \to X$ is homotopic to the constant mapping $const_{x_0} : X \to x_0$ that takes all X to the point $x_0 \in X$.

Membership of $X \in C^k$ is equivalent to the following:

For any simplex T of dimension ≥ 1, any continuous map has a continuous extension $F: T \to X$ such that any base set O of the space X containing $f(A)$ also contains $F(T)$, where A is the union of an arbitrary set of faces of the simplex T of dimension ≥ 1 , including all one-dimensional.

Now we need an ℓ_2 Hilbert space, the points of this space are all possible sequences of $x = \{x_k : k \in N\}$ real numbers satisfying the $\sum_{k=1}^{\infty} x_k^2 < \infty$ condition; and the norm (metric) of an element $x \in \ell_2$ is given by the formula

$$\|x\| = (\sum_{k=1}^{\infty} x_k^2)^{\frac{1}{2}} \tag{5}$$

The following takes place

Theorem [3]. The Hilbert space ℓ_2 considered as a topological space is homomorphic to a countable infinite power R^{ω} of the real line.

An n-dimensional Euclidean space R^n can be defined as the subspace of a Hilbert space consisting of those points $x = \{x_k : k \in N\}$ from ℓ_2 for which $x_k = 0$ for all $k > n$.

A system of points $x^i = \{x_k^i\}, i = 0,1,2,...,m$, from ℓ_2 is called independent if their linear combination, $\mu_0 x^0 + \mu_1 x^0 + ... + \mu_m x^m$, where $\mu_0 + \mu_1 + ... + \mu_m = 0$ is equal to zero, only when all $\mu_i = 0$.

If the system of points $x^i, i = 0,1,...,m$ is independent, then the collection of points $x \in \ell_2$ representable in the form $x = t_0 x^0 + t_1 x^1 + ... + t_m x^m$, where $t_i \geq 0$ and $t_0 + t_1 + ... + t_m = 1$ is called a (closed) m–dimensional simplex with vertices $x^0, x^1, x^2, ...x^m$ is denoted by $T(x^0, x^1, x^2, ...x^m)$ the coefficients $t_0, t_1, t_2, ...t_m$ uniquely determined by the point x and uniquely determining this point are called its barycentric, (in a barycentric coordinate system consisting of points $x^0, x^1, x^2, ...x^m$). If a $T-$(closed) m-dimensional simplex with vertices $x^0, x^1, x^2, ...x^m$, then the set of all points $x \in T$, all barycentric coordinates of which in the system $x^0, x^1, x^2, ...x^m$ are positive, is called an open m-dimensional simplex (with the same vertices as T) and denoted by $< T >$. The difference $T \backslash < T > = bdT$ is called the boundary of the simplex T. By the face of the simplex T we mean the simplex T, all of whose vertices are the vertices of a given simplex T. A set (lying in a Hilbert space) represented as a finite (or locally finite) union of simplices is called a polyhedron. The union of all at most $k-$dimensional simplices of the polyhedron P is called its $k-$dimensional skeleton and is denoted by $\mathrm{P}^{(k)}$. In particular, the zero-dimensional skeleton $\mathrm{P}^{(0)}$ of the polyhedron P coincides with the set of all its vertices [3-7].

Definition [3]. A mapping $p: X \to Y$ is said to be locally soft if, for any separable metrizable space B, any of its closed subspaces A, and any two mappings $\alpha: A \to X$ and $\rho: B \to Y$ such that $p_\alpha = p|_A$ there is a mapping $\gamma: U \to X$ defined on some open neighborhood U of the set A in B such that $\gamma/_A = \alpha$ and $p\gamma = \rho/u$, those. there is a diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ \alpha\uparrow & \nwarrow^{\gamma} & \uparrow\rho \\ A \subset & U \subset & B \end{array} \tag{6}$$

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ \alpha\uparrow & \nwarrow^{\gamma} & \uparrow\rho \\ A & \subset & B \end{array} \tag{7}$$

If, under the indicated conditions, the neighborhood U can always be assumed to coincide with the entire space B, then the mapping P will be called soft.

Let us agree to say that a space (recall that we consider only separable metrizable spaces) is an absolute (neighborhood) extensor if the constant mapping of this space is a (locally) soft mapping. The class of all absolute (neighborhood) extensors is denoted by $A(N)E$.

The definition of the functor $\mathrm{Set}(X)$ and the structure of its open sets, as well as the continuity of open mappings, imply the following [8-10]:

Theorem [9]. Let $f: X \to Y$ be a continuous open mapping between topological spaces X and Y. Then the mapping $\mathrm{Set}(f): \mathrm{Set}(X) \to \mathrm{Set}(Y)$ is also open.

We say that a category K is given if a class of objects O and a class of morphisms $Mor(A,B)$ elements, category K, are given, and

1. For every pair of objects (A,B) from K, a set $Mor(A,B)$ is given, called a morphism A into B;
2. For each triple of objects (A,B,C) from K, a mapping is given

ISSN: 2237-0722
Vol. 11 No. 2 (2021)
Received: 10.03.2021 – Accepted: 12.04.2021

$\mu : Mor(A,B) \times Mor(B,C) \to Mor(A,C)$ ($\mu(u,v)$ image of a pair

$(u,v) \in Mor(A,B) \times Mor(B,C)$ is denoted by $v \cdot u$ and is called the composition of the morphisms u and v;

3. The sets $Mor(A,B)$ and the composition of morphisms satisfy the following axioms:

a. The composition is associative, i.e. for each triple of morphisms u, v, ω, the equality $\omega \cdot (u \cdot v) = (\omega \cdot u) \cdot v$ holds;
b. For each object A from K there is a morphism $1_A : A \to A$, called the identity morphism of the object A, such $1_A \cdot u = u$ and $v \cdot 1_A = v$ for any morphisms $u \in Mor(B,A)$ and $v \in Mor(A,B)$;
c. all pairs (A,B) and (A',B') are different, then the intersection of the sets $Mor(A,B)$ and $Mor(A',B')$ is empty.

An important example of a category is the category TOP of all topological spaces and their continuous mappings. The set $Mor(X,Y)$ for any two topological spaces is in this case the simple set of all continuous mappings from X to Y. Composition is understood as the usual composition of mappings.

Quite similarly, a category is also formed by collections of different topological spaces with an appropriate choice of the properties characterizing them. For example, a category is the class of all compact spaces and their continuous mappings into themselves, etc. [11-16].

Example. Category TOP_Y.

Another example of a category is the category TOP_Y. Objects of this category are continuous mappings into a given topological space Y. The morphism connecting for an object $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ is a mapping $g : X_1 \to X_2$ such that $f_2 g = f_1$. To be extremely precise, the morphism is a triangular commutative diagram

$$\begin{tikzcd} X_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & Y & \end{tikzcd} \qquad (8)$$

Identity mapping of object $f: X \to Y$ is identity mapping id_X.

The product of a family of objects in a category TOP_Y is defined as follows: Let a family of mappings $f_\alpha : X_\alpha \to Y$, $\alpha \in A$ be given. In a topological product $Y^A = \prod\{Y_\alpha : \alpha \in A\}$, where $Y_\alpha = Y$, for each $\alpha \in A$, consider the diagonal

$$\Delta(Y^A) = \{\{y_\alpha : \alpha \in A\} \in Y^A : y_\alpha = y_\beta, \forall \alpha, \beta \in A\}.$$

We denote by X the complete preimage of the diagonal $\Delta(Y^A)$ with respect to the mapping $\prod\{Y_\alpha : \alpha \in A\}$, and by f_A the restriction of this mapping to the set X. The natural homeomorphism of the diagonal $\Delta(Y^A)$ onto the space Y is denoted by δ_A. We denote the composition $\delta_A f_A$ by f. The space X will be called the fan product of the spaces X_α with respect to the mappings f_α. If Y is one-point, then the fan product, as is easy to see, coincides with the usual one. The mappings f are called the fibrewise product of the mappings f_α. It can be verified that the fibrous product f of the mappings f_α is the product of objects f_α in the category TOP_Y. Let us also agree to denote by φ_α the restrictions of the projections $\pi_\alpha : \Pi\{X_\alpha : \alpha :\in A\} \to X_\alpha$ on X and call them projections of the fan product. It is easy to see that the equality $f = f_\alpha \varphi_\alpha$ holds for any $\alpha \in A$.

Each topological space can be identified with its mapping into some fixed one-point space and, therefore, the category TOP is a subcategory of the category TOP_Y.

Let $\mathscr{G} = (V, \mathrm{M})$ and $\mathscr{G}' = (V', \mathrm{M})$ be two categories. A mapping $\mathscr{F} : \mathscr{G} \to \mathscr{G}'$ that transforms objects into objects and morphisms into morphisms is called a covariant (contravariant) functor from category $\mathscr{G}$ to category $\mathscr{G}'$ if:

1) for any morphism $f: X \to Y$ from the category $\mathscr{G}$, the morphism $\mathscr{F}(f)$ acts from $\mathscr{F}(X)$ to $\mathscr{F}(Y)$ (from $\mathscr{F}(X)$ to $\mathscr{F}(Y)$);
2) $\mathscr{F}(id_X) = id_{\mathscr{F}(X)}$ for every $X \in V$;
3) $\mathscr{F}(f \circ g) = \mathscr{F}(f) \circ \mathscr{F}(g)$ $(\mathscr{F}(g) \circ \mathscr{F}(f))$;

Let $\mathscr{F}_i : \mathscr{G} \to \mathscr{G}'$, $i = \overline{1,2}$ be two covariant functors from a category $\mathscr{G} = (V, \mathrm{M})$ to a category $\mathscr{G}' = (V', \mathrm{M})$. A family of morphisms $\Phi = \{f_X : \mathscr{F}_1(X) \to \mathscr{F}_2(X), x \in V\} \subset V'$ is called a natural transformation of a functor $\mathscr{F}_1$ into a functor $\mathscr{F}_2$ if, for every morphism $f : X \to Y$ of a category $\mathscr{G}$, the diagram

$$\begin{tikzcd} \mathscr{F}_1(X) \arrow[r, "f_X"] \arrow[d, "\mathscr{F}_1(f)"'] & \mathscr{F}_2(X) \arrow[d, "\mathscr{F}_2(X)"] \\ \mathscr{F}_1(Y) \arrow[r, "f_Y"] & \mathscr{F}_1(Y) \end{tikzcd} \tag{9}$$

Let $\mathscr{F}_1, \mathscr{F}_2$ be functors acting from the category V into itself. A functor $\mathscr{F}_1$ is called a subfunctor (overfunctor) of the function $\mathscr{F}_2$ if there exists a natural transformation $\Phi = \{f_X\} : \mathscr{F}_1 \to \mathscr{F}_2$ such that every mapping f_X is an embedding (epimorphism). In the second case, the functor $\mathscr{F}_2$ is called the functor-functors of the functor $\mathscr{F}_1$.

Note that the essential functor Id and the functors $\exp$, $\exp^c$, $\exp^{ccon}$ are covariant subfunctors of the functor Set on the category of Top-topological spaces and continuous mappings into itself. The definition of a covariant functor and the definition of morphisms $\mathrm{Set}(f)$ on the category Top of topological spaces and continuous mappings into itself imply the following.

Theorem 1. a) The functor Set is a covariant functor in the category Top;

b) The functor $\mathrm{Set} : Top \to Top$ preserves the layers of a continuous mapping, that is, $(\mathrm{Set}(f))^{-1}(f(A)) = \mathrm{Set}(f^{-1}(A))$.

Let X be an infinite topological T_1--space. We denote by $\mathrm{Set}_{\chi_0}(X)$ the family of all finite and countable subsets of the space X, that is,

$$\mathrm{Set}_{\chi_0}(X) = \{F : F \subseteq X; |F| \leq \chi_0\};$$

$$\exp_n(X) = \{F : F \subseteq X; |F| \leq n\};$$

$$\exp_\omega(X) = \bigcup_{n=1}^{\infty} \exp_n(X);$$

from here

$$\exp_\omega(X) \subseteq \mathrm{Set}_{\chi_0}(X),\ \mathrm{Set}_{\chi_0}(X) \subseteq \mathrm{Set}(X).$$

ISSN: 2237-0722
Vol. 11 No. 2 (2021)
Received: 10.03.2021 – Accepted: 12.04.2021

For any map $f: X \to Y$, the fiberwise exponent $\overline{f}_f : X \to Y_1$ where $X_f = \{F \in \mathrm{Set}(X) : F \subset f^{-1}(y), y \in Y\}$ is defined, so that $\overline{f}(F) = y$ if $F \subset f^{-1}(y)$.

Consider the commutative diagram

$$\begin{array}{ccc} Z_0 & \xrightarrow{g} & X \\ \downarrow & k\square & \downarrow f \\ \mathrm{Z} & \xrightarrow[n]{} & \mathrm{Y} \end{array} \qquad (10)$$

where Z_0 is a closed subset of the compact set.

(*): The problem of the existence of an extension $k: Z \to X$ of a mapping g such that $f \circ k = h$.

The mapping f is called $n-$soft, $n = 0,1,2,...$ if problem (*) has a solution whenever $\dim\square \le n$; $\infty-$soft mappings are simply called soft mappings.

A subset A (not necessarily closed) of a space X is called $C-$embedded if every continuous function $f: A \to R$ (respectively, $f: A \to I = [0,1]$) is continuous to all X.

A space Y is called contractible (for short, $Y \in C$) if there exists a homotopy $H(y,t): Y \times I \to Y$ such that $H(y,0) = id_Y$ and $H(y,1)$ are trivially locally contractible (for short, $Y \in LC$) if any point $y \in Y$ and any of its neighborhoods U has a smallest neighborhood $V \subseteq U$ such that the written embedding is homotopy trivially.

Theorem 2. The functor Set preserves the contractibility of topological spaces.

Proof. Let X be a contractible space and $H(x,t): X \times [0,1] \to X$ a homotopy connecting the mappings $H(x,1) = id_X = h_0$ and $H(x,1) = 1_{x_0} = h_1$. An embedding $i_t : X \times \{t\} \to X \times I$ for each $t \in [0,1]$ defines an embedding $Set(i_t): Set(X \times \{t\}) \to Set(X \times [0,1])$. But the space $Set(X \times \{t\})$ is naturally homeomorphic to the space $Set(X \times \{t\})$. Therefore, a natural embedding $Set(X) \times I \to Set(X \times [0,1])$ is defined. Then the restriction $Set(H(x,t))|_{Set(X)\times I}$ of the mapping $Set(H(x,t))$ is a continuous homotopy connecting the mappings $Set(h_0)$ and $Set(h_1)$, that is, it is the desired homotopy. Hence the space $Set(X)$ is contractible. Theorem 2 is proved.

If X is nested in Y, then $\mathrm{Set}(X)$ is also nested in $\mathrm{Set}(Y)$.

ISSN: 2237-0722
Vol. 11 No. 2 (2021)
Received: 10.03.2021 – Accepted: 12.04.2021

A set A of a space X is called a retract X if there exists a continuous mapping $r: X \to A$ such that $r(a) = a$ is true for any $a \in A$, that is, there exists a continuous map of the space X onto the subspace A leaving the points of the set A fixed.

Proposition [5]. A subspace A is a retract of the space X if and only if, for any space Y, each map $g: A \to Y$ admits an extension to the space X.

If a set A is everywhere dense in X, then $\mathrm{Set}(A)$ is also everywhere dense in $Set(X)$.

Question: if X is a T_i space, then $\mathrm{Set}(X)$ is also a T_i space?

Let X be a topological $T_1 -$space

$\mathrm{Set}(X) \cdot i_X(X) \subset \mathrm{Set}(X)$, $i_X(X)$ $C-$nested, in $\mathrm{Set}(X)$, i.e. $f: i_X(X) \to R$ is continuous, $\exists$ $f: \mathrm{Set}(X) \to R$ that $f|_{i_X(X)} = f$.

$O(u_1, u_2, ..., u_k) = \{A : A \in \mathrm{Set}(X), A \subset \bigcup_{i=1}^{k} u_i, A \bigcap u_i \neq \varnothing, i = \overline{1,k}\}$ are the basic sets of the space $\mathrm{Set}(X)$. Now we consider the complement of the basic set in the space $\mathrm{Set}(X)$, that is, which sets are closed sets in $\mathrm{Set}(X)$.

On the other hand:

$$O(u_1, u_2, ..., u_k) = \{A : A \in \mathrm{Set}(X), A \subset \bigcup_{i=1}^{k} u_i\} \bigcap (\bigcap_{i=1}^{k} \{A : A \in \mathrm{Set}(X), A \bigcap u_i \neq \varnothing\}) \quad (11)$$

$$\mathrm{Set}(X) \setminus O(u_1, u_2, ..., u_k) = (\mathrm{Set}X \setminus \{A : A \in \mathrm{Set}(X); A \subset \bigcup_{i=1}^{k} u_i\}) \bigcup$$

$$\bigcup (\mathrm{Set}X \setminus \bigcap_{i=1}^{k} \{A : A \in \mathrm{Set}X, A \bigcap u_i \neq \varnothing\}) = \mathrm{Set}X \setminus \{A : A \in \mathrm{Set}X, A \subset \bigcup_{i=1}^{k} u_i\} \bigcup \quad (12)$$

$$\bigcup (\bigcup_{i=1}^{k} \mathrm{Set}(X) \setminus \{A : A \in \mathrm{Set}X, A \bigcap u_i \neq \varnothing\}).$$

Hence, this set consists of the union of the following sets:

a. $\mathrm{Set}X \setminus \{A : A \in \mathrm{Set}X : A \subset \bigcup_{i=1}^{k} u_i\}$; (13)

b. $\mathrm{Set}X \setminus \{A : A \in \mathrm{Set}X : A \bigcap u_i \neq \varnothing\}$;

Now, if we expand, we get

$$\mathrm{Set}X \setminus \{A: A\in \mathrm{Set}X, A\subset \bigcup_{i=1}^{k} u_i\} = \{A: A\in \mathrm{Set}X, A\subset X\setminus \bigcup_{i=1}^{k} u_i = \bigcap_{i=1}^{k}(X\setminus u_i)\} \quad (14)$$

$$\mathrm{Set}X \setminus \{A: A\in \mathrm{Set}X, A\bigcap u_i \neq \varnothing\} = \{A: A\in \mathrm{Set}X, A\bigcap u_i \neq \varnothing\} == \{A: A\in \mathrm{Set}X, A\subset X\setminus u_i\}$$

It turns out that if $u_1, u_2, ..., u_k$ are open, then the sets

$$\{A: A\in \mathrm{Set}X, A\subset \bigcup_{i=1}^{k} u_i\} = \mathrm{Set}(\bigcup_{i=1}^{k} u_i, X), \quad (15)$$

$$\{A: A\in \mathrm{Set}X, A\bigcap u_i \neq \varnothing\} = \mathrm{Set}\,X \setminus \mathrm{Set}(X\setminus u, X)$$

are open in the space of nonempty subsets by the definition of the exponential set $\mathrm{Set}(X)$.

On the other hand, the elements of the prebase that we specified when defining the exponential topology have this form:

$$\mathrm{Set}(u, X) = O(u),\ \mathrm{Set}(X)\setminus \mathrm{Set}(X\setminus u, X) = O(u, X).$$

Thus, the families of all sets of the form $O < u_1, u_2, ..., u_k >$, where the sets $u_1, u_2, ..., u_k$ are open in the space X, are the base of the exponential topology in the space $\mathrm{Set}(X)$.

Condition [1]. $LC^{\#}$: There is a base of neighborhoods satisfying Condition $C^{\#}$.

Theorem [1]. If a topological space Z has the property $LC^{\#}$, then for any of its open coverings γ there is an open cover ω of it such that for every polytope P_C by the Whitehead topology every ω-realization of $f: A \to Z$ (where an $A-$subpolytop containing all edges from *P*) has extensions $ext: P \to$, which is a $\gamma -$implementation.

Condition. $C^{\#}$: For any simplex T of dimension ≥ 1, any continuous mapping has a continuous extension $F: T \to Z$ such that any base set O_τ of the space Z containing $f(A)$ contains $F(T)$.

Condition equivalent to $C^{\#}$: Any continuous mapping $f: S^n \to Z, n\geq 1$, has a continuous extension $F: Q^{n+1} \to Z$ to the ball bounded by the sphere S^n such that if $f(S^n)\subset O_\tau$, then $f(Q^{n+1})\subset O_\tau$.

For a topological space X, here denotes one of the following spaces: $SetX$, $Comp$, $ConnX$, $ContX$ for arbitrary X, and $\exp X$, $\exp X \bigcap CompX$, $\exp X \bigcap ConnX$, $\exp X \bigcap ContX$ (for regular X) where $\exp X$ is the family of all non-empty closed subsets,

ISSN: 2237-0722
Vol. 11 No. 2 (2021)
Received: 10.03.2021 – Accepted: 12.04.2021

$SetX$ is the family of all non-empty subsets, $CompX$ -family of all non-empty compact subsets, $ConnX$ -family of all non-empty connected subsets, $ContX$ -family of all non-empty connected compact subsets of X [17].

Proposition. The functor Set preserves retractions.

Proof. Let X be a topological space $A \subset X$ and $r : X \to A$a continuous retraction. Then

The mapping $Set(r) : Set(X) \to Set(A)$ is also continuous [1]. It is known that $Set(A) \subset Set(X)$ and the retraction $r : X \to A$ leave the points of the subspace A movable. Consequently, under the mapping $Set(r)$, the points of the space $Set(A)$ remain fixed, since the functor Set preserves points and empty sets. Hence, $Set(r)$ is a retraction, i.e. $Set(A)$ is a retract of $Set(X)$. The proposition is proved.

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